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Commuting Outer Inverse-Based Solutions to the Yang–Baxter-like Matrix Equation

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Abstract: This paper investigates new solution sets for the Yang–Baxter-like (YB-like) matrix equation involving constant entries or rational functional entries over complex numbers. Towards this aim, first, we introduce and characterize an essential class of generalized outer inverses (termed as $\{2,5\}$-inverses) of a matrix, which commute with it. This class of $\{2,5\}$-inverses is defined based on resolving appropriate matrix equations and inner inverses. In general, solutions to such matrix equations represent optimization problems and require the minimization of corresponding matrix norms. We decided to analytically extend the obtained results to the derivation of explicit formulae for solving the YB-like matrix equation. Furthermore, algorithms for computing the solutions are developed corresponding to the suggested methods in some computer algebra systems. The main features of the proposed approach are highlighted and illustrated by numerical experiments.

Keywords: Yang–Baxter-like matrix equation; outer inverse; Moore–Penrose inverse; idempotent matrices; computer algebra

MSC: 15A09; 15A24; 68W30

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1. Introduction and Literature Review

Given a square matrix A; we are concerned about finding the unknown matrix X which fulfills the matrix equation

$$XAX = AXA. \tag{1}$$

This equation is called the Yang–Baxter-like (YB-like, for short) matrix equation. If A is singular (resp. nonsingular), we call (1) the singular (resp. nonsingular) YB-like matrix equation. Furthermore, if the entries of A are constants (resp. multivariate rational functions with coefficients) over the field of complex numbers $\mathbb{C}$, then Equation (1) is said to be the constant (resp. rational) YB-like matrix equation.

Equation (1) possesses a similar format to the famous Yang–Baxter equation, first introduced by Yang [1] in 1967 and then by Baxter [2] independently in 1972, in the field of statistical mechanics. The classic Yang–Baxter equation has been a hot research area in science and engineering applications, closely related to various mathematical subjects, such as knot theory [3], braid groups [4], statistical mechanics [5], and quantum research [6]. So, it is necessary to find partial or general solutions of (1) from the viewpoint of matrix theory. The YB-like matrix equation is identifiable as the star-triangle-like equation in statistical mechanics ([7], [Part III]) and [8].

Notice that (1) is a quadratic matrix equation with at least two (trivial) solutions, $X = A$ and $X = 0$, but its nonlinearity makes it challenging to solve: the problem of calculating a nontrivial solution requires solving a system of n^2 quadratic equations with n^2 variables, which is a complex task.

Many direct methods have recently been constructed to find several classes of solutions to (1) and most of them are based on the structure of A; see, e.g., [9–24] and references therein. To illustrate it further, all solutions were investigated in [9] for the matrix $A = I - uv^{\mathrm{T}}$ such that $v^{\mathrm{T}}u \neq 0$, where u and v are n-dimensional vectors. In [10], commuting solutions have been located for the situation where A has some particular Jordan forms. Solutions to (1) for some types of Jordan canonical form of A were suggested in [11]. All commuting solutions for a diagonalizable matrix and non-commuting solutions for a Householder matrix are discovered in [12]. All solutions to (1) were found in [13–15] when A is an idempotent and a rank-one matrix, respectively. Spectral solutions were studied in [16,17]. It was shown in [18] that any semisimple eigenvalue of a matrix A gives rise to infinitely many solutions. This result was extended in [19] to the matrix A having a non-semisimple eigenvalue with at least 1×1 Jordan block. The complete solution set can be attained from [20–22] for the matrix A of rank two. All the commuting solutions were found for the matrix $A = I - PQ^{\mathrm{T}}$, where P and Q are two $n \times 2$ matrices of full column rank and $\det(Q^{\mathrm{T}}P) \neq 0$. All solutions that commute with A were identified in [23] provided $A^3 = 0$. In [24], all the solutions were observed for A such that $A^2 = 0$ and has a rank equal to one or two.

However, by increasing the dimension of the input matrix A, direct methods cannot be used in practice for solving the Equation (1) due to considerable cost in both time and space requirements. This observation has led some analysts to suggest and rely on numerical methods to discover solutions. Such solutions were obtained in [25] using the classic Brouwer fixed point theorem for a nonsingular quasi-stochastic matrix A such that A^{-1} is stochastic. The authors of [26] proposed iterative methods for calculating commuting solutions via the mean ergodic theorem for the diagonalizable matrix A. Some iterative methods have been presented in [27] for an arbitrary matrix A. An iterative method based on the Hermitian and skew-Hermitian splitting of the matrix A was introduced in [28].

Zeroing neural network (ZNN) dynamical system approach has been exploited in solving the time-varying Yang–Baxter matrix equation $X(t)A(t)X(t) = A(t)X(t)A(t)$, where the given $A(t)$ and the unknown $X(t)$ are real time-varying square matrices. Various ZNN dynamical systems were proposed in [29–31].

The rank optimization problem $\min_{AXA=B} \operatorname{rank}(B - XAX)$ related to the YB matrix equation was considered in [32].

After all, the computation of the solutions to the rational YB-like equation in symbolic implementation has not been investigated so far. The symbolic calculation is an essential area of computer algebra and scientific computing. Moreover, there has not been any efficient algorithm for solving YB-like equation if the matrix A is arbitrary and with entries given as rational functions with an arbitrary number of variables with coefficients over complex numbers. This paper contributes to resolving these issues. Recall that in [33], the authors have developed effective formulae for calculating infinitely many solutions using finite-precision arithmetic. Nevertheless, those methods do not perform well when A is an ill-conditioned or a matrix with multivariate rational functional entries.

The global organization is based on the following sections. Main generalized inverses and corresponding matrix equations that define them are surveyed in Section 2. Some notations, notions, and motivations are also introduced and discussed therein. Next, in Section 3, we provide a theoretical basis for determining a specific variety of {2,5} generalized inverses in terms of inner inverses. The required inner inverses can be generated as solutions of an appropriate couple of linear matrix equations. Then in Section 4, we set up a correlation between the explicit solutions to the YB-like matrix equation and {2,5}-inverses of some appropriate matrix. Algorithms for solving the matrix Equation (1) are developed based on introduced results in the previous section. These algorithms are easily implementable in the programming language `MATHEMATICA`. Numerical experiments are given in Section 5 to support the claims given in this work. Finally, the conclusions of this paper will be drawn in Section 6.

2. Preliminaries and Motivation

Let $\mathbb{C}(\mathbf{z})$ be the set of multivariate rational functions with complex coefficients in the unknown variables $\mathbf{z} = (z_1, \ldots, z_p)$. As usual, $\mathbb{C}^{m\times n}$ (resp. $\mathbb{C}(\mathbf{z})^{m\times n}$) denotes the set of $m \times n$ matrices over $\mathbb{C}$ (resp. over $\mathbb{C}(\mathbf{z})$), while $\mathbb{C}_r^{m\times n}$ (resp. $\mathbb{C}(\mathbf{z})_r^{m\times n}$) stands for the subset of $\mathbb{C}^{m\times n}$ (resp. of $\mathbb{C}(\mathbf{z})^{m\times n}$) which includes matrices of rank r. The symbol I stands for the identity matrix of an appropriate order. By M^*, $\mathrm{rank}(M)$, $\mathcal{R}(M)$ and $\mathcal{N}(M)$, we mean the conjugate-transpose, the rank, the range and the null space of a matrix $M \in \mathbb{C}(\mathbf{z})^{m\times n}$, respectively. The index of a square matrix M is defined as $\mathrm{ind}(M) = \min\{k|\mathcal{R}(M^k) = \mathcal{R}(M^{k+1})\}$. The following matrix equations

$$\begin{array}{lll} (1)\ MXM = M & (2)\ XMX = X & (3)\ MX = (MX)^* \\ (4)\ XM = (XM)^* & (1^k)\ M^{k+1}X = M^k,\ k \geq \mathrm{ind}(M) & (5)\ MX = XM \end{array}$$

define different classes of generalized inverses of a nonzero matrix $M \in \mathbb{C}(\mathbf{z})^{m\times n}$; see [34–38]. In fact, if $\Gamma \subseteq \{1, 2, 3, 4, 1^k, 5\}$, then a complex matrix X is called a Γ-inverse of M if X satisfies equation (n), for each $n \in \Gamma$. The notation $M\{i, j, \ldots, k\}$ stands for the set of all Γ-inverses if $\Gamma = \{i, j, \ldots, k\}$. Any matrix from $M\{i, j, \ldots, k\}$ is always denoted by $M^{(i,j,\ldots,k)}$. Particularly, $M^{(1,2,3,4)} = M^\dagger$, called the Moore–Penrose inverse of M, which always exists and is unique. Furthermore, there is a unique inverse $M^{(2,5,1^k)}$ of a square matrix M called the Drazin inverse, and M^{D} is its label. The Drazin inverse coincides with the group inverse $X = M^\#$ if $\mathrm{ind}(A) = 1$.

A selected $X \in M\{i, j, \ldots, k\}$ which fulfils $\mathcal{R}(X) = \mathcal{R}(E)$ as well as $\mathcal{N}(X) = \mathcal{N}(F)$ will be termed as $M^{(i,j,\ldots,k)}_{\mathcal{R}(E),\mathcal{N}(F)}$. A matrix is said to be an outer generalized inverse of M if it belongs to $M\{2\}$. The fundamental result that describes the existence of outer inverse with prescribed range and null space of $M \in \mathbb{C}_r^{m\times n}$ is restated in Lemma 1.

Lemma 1 ([34], Theorem 2.14)**.** *Let $M \in \mathbb{C}_r^{m\times n}$, let T be a subspace of $\mathbb{C}^n$ of dimension $t \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - t$. Then, M has a $\{2\}$-inverse (or outer inverse) X such that $R(X) = T$ and $N(X) = S$ if and only if $MT \bigoplus S = \mathbb{C}^m$, in which case X is unique and is denoted by $M^{(2)}_{T,S}$.*

There exist a number of representations for outer inverses with determined range and null space in the literature [39–45]. Now we revisit an important lemma for determining $M^{(2)}_{T,S}$.

Lemma 2 ([42])**.** *For the same $M, T,$ and S as in Lemma 1, the $\{2\}$-inverse $M^{(2)}_{T,S}$ exists if and only if there exists $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$, $\mathcal{N}(G) = S$ and $\mathrm{rank}(GMG) = \mathrm{rank}(G)$. Furthermore,*

$$M^{(2)}_{T,S} = (GM)^\# G = G(MG)^\# = G(GMG)^{(1)} G.$$

Our main intention in this paper is the development of algorithms for finding the solutions to the rational YB-like matrix equation $XAX = AXA$. These algorithms are based on newly derived solution representations to the desired matrix equation. Towards this aim, we generate a unique approach with the help of the class of {2,5}-inverses of a nonzero matrix $M = \alpha I + \beta A$, for appropriate $\alpha, \beta \in \mathbb{C}$. Such matrix inverses will be named as commuting outer inverses of M. Developed algorithms are based on solving a suitable system of linear matrix equations under exact rank conditions. The underlying matrix equations are considered as minimization problems and can be solved using various methods. We use exact and numerical solutions to these matrix equations in a computer algebra system. In further steps, commuting outer inverses of M are used to define a relevant projector P and two appropriate choices of a matrix B. Each choice of B is finally used in defining a collection of solutions to the YB-like matrix equation. This central goal is developed through the following primary outcomes of this research.

(a) Several equivalent characterizations and initiated representations of $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ are given.
(b) Necessary and sufficient conditions when $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ becomes $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ are investigated.
(c) Proposed results about the requirement $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} \in M\{2,5\}$ as well as computational procedures for obtaining $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ are applied with the aim of deriving explicit formulae to solve the YB-like Equations (1).
(d) Algorithms for solving YB–like matrix equation with constant entries or entries given as rational functions with several variables are presented.
(e) Implementation of the proposed algorithms in the MATHEMATICA computer Algebra system is developed, and illustrative examples are executed.

3. Existence, Characterizations and Representations of {2,5}-Inverses

This section will examine some crucial properties of {2,5}-inverses with the prescribed range and null space of a square matrix M.

The forthcoming Theorem 1 and Corollary 1 provide equivalent conditions for the existence and representations of $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ as well as $M^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)}$. Domain of obtained representations are complex square matrices and utilize solutions of a particular pair of linear matrix equations. Consequently, some new relationships are established between solutions to the linear matrix equations and obtaining {2,5}-inverses with determined range and null space.

Theorem 1. *Let* $M \in \mathbb{C}(\mathbf{z})^{n\times n}$, $E \in \mathbb{C}(\mathbf{z})^{n\times k}$, *and* $F \in \mathbb{C}(\mathbf{z})^{l\times n}$.

(a) *The subsequent statements are mutually equivalent:*

(i) $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ *exists;*
(ii) *There exists* $U \in \mathbb{C}(\mathbf{z})^{k\times l}$ *satisfying* $E = MEUFE$, *and* $F = FEUFM$;
(iii) *There exist* $U, V \in \mathbb{C}(\mathbf{z})^{k\times l}$ *satisfying* $E = MEUFE$, *and* $F = FEVFM$;
(iv) *There exist* $U \in \mathbb{C}(\mathbf{z})^{k\times n}$ *and* $V \in \mathbb{C}(\mathbf{z})^{n\times l}$ *satisfying* $E = MEUE$, $F = FVFM$, *and* $EU = VF$;
(v) *There exist* $U \in \mathbb{C}(\mathbf{z})^{k\times n}$ *and* $V \in \mathbb{C}(\mathbf{z})^{n\times l}$ *satisfying* $E = MVFE$, $F = FEUM$, *and* $EU = VF$;
(vi) $E = ME(FME)^{(1)}FE$, *and* $F = FE(FME)^{(1)}FM$, *for some (equivalently every) inner inverse* $(FME)^{(1)} \in (FME)\{1\}$.

(b) *If an arbitrary of the statements* (i)–(vi) *is valid, then*

$$M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(FME)^{(1)}F = EUF,$$

for arbitrary $(FME)^{(1)} \in (FME)\{1\}$ *and an arbitrary* $U \in \mathbb{C}(\mathbf{z})^{k\times l}$ *satisfying* $E = MEUFE$, *and* $F = FEUFM$.

Proof. (a) (i) ⇒ (ii): Let $X = M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ be such that $XMX = X$, $XM = MX$, $\mathcal{R}(X) = \mathcal{R}(E)$ and $\mathcal{N}(X) = \mathcal{N}(F)$. Then there exists $U \in \mathbb{C}(\mathbf{z})^{k\times l}$ which determines X by $X = EUF$. In addition, E and F fulfill the conditions $E = XW$ and $F = VX$, for some $W \in \mathbb{C}(\mathbf{z})^{n\times k}$, $V \in \mathbb{C}(\mathbf{z})^{l\times n}$. This further implies

$$E = XW = (XMX)W = XM(XW) = XM(E) = (MX)E = M(X)E = MEUFE,$$
$$F = VX = V(XMX) = (VX)MX = (F)MX = F(XM) = F(X)M = FEUFM.$$

(ii) $\Rightarrow$ (iii): This implication is clear.

(iii) $\Rightarrow$ (vi): Assume the existence of $U, V \in \mathbb{C}(\mathbf{z})^{k\times l}$ such that $E = MEUFE$ and $F = FEVFM$. In that case

$$E = MEUFE = MEU(F)E = MEU(FEVFM)E = (MEUFE)VFME = EVFME.$$

It further yields

$$\begin{aligned} E &= EVFME = EV(FME) \\ &= EV\Big(FME(FME)^{(1)}FME\Big) = (EVFME)(FME)^{(1)}FME \\ &= E(FME)^{(1)}FME. \end{aligned}$$

Hence,

$$\begin{aligned} E &= MEUFE = M(E)UFE = M(E(FME)^{(1)}FME)UFE \\ &= ME(FME)^{(1)}F(MEUFE) = ME(FME)^{(1)}FE. \end{aligned}$$

Similarly we can prove $F = FE(FME)^{(1)}FM$.

(vi) $\Rightarrow$ (i): If (vi) holds, one obtains

$$\begin{aligned} E &= ME(FME)^{(1)}FE = ME(FME)^{(1)}(F)E \\ &= ME(FME)^{(1)}(FE(FME)^{(1)}FM)E = (ME(FME)^{(1)}FE)(FME)^{(1)}FME \\ &= E(FME)^{(1)}FME. \end{aligned}$$

Thus, $E = E(FME)^{(1)}FME$. Similarly, $F = FME(FME)^{(1)}F$. Therefore, by ([41], [Theorem 6]) and ([43], [Corollary 2.5]), $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ exists and $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(FME)^{(1)}F$. Notice that

$$\begin{aligned} M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M &= (E(FME)^{(1)}F)M = (E)(FME)^{(1)}FM \\ &= (ME(FME)^{(1)}FE)(FME)^{(1)}FM = ME(FME)^{(1)}(FE(FME)^{(1)}FM) \\ &= ME(FME)^{(1)}(F) = M(E(FME)^{(1)}F) \\ &= MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}. \end{aligned}$$

(ii) $\Rightarrow$ (iv): Suppose existence of $U \in \mathbb{C}(\mathbf{z})^{k\times l}$ satisfying $E = MEUFE$ and $F = FEUFM$. Such assumptions initiate

$$\begin{aligned} E &= MEUFE = ME(UF)E, \\ F &= FEUFM = F(EU)FM, \\ E(UF) &= (EU)F, \end{aligned}$$

which confirms (iv).

(iv) $\Leftrightarrow$ (v): This equivalence is evident.

(v) $\Rightarrow$ (i): Let the equations $E = MVFE$, $F = FEUM$ and $EU = VF$ are fulfilled for some $U \in \mathbb{C}(\mathbf{z})^{k\times n}$ and $V \in \mathbb{C}(\mathbf{z})^{n\times l}$. Consider $X = EU = VF$. Such conditions initiate

$$MVF = MV(F) = MV(FEUM) = (MVFE)UM = EUM,$$

which gives $MX = XM$. Also

$$XMX = (EU)M(EU) = (EUM)EU = (MVF)EU = (MVFE)U = EU = X.$$

Now $E = MVFE = (MVF)E = (EUM)E = (EU)ME = XME$, which implies $\mathcal{R}(X) = \mathcal{R}(E)$. In addition, based on $F = FEUM = F(EUM) = F(MVF) = FM(VF) = FMX$, it follows $\mathcal{N}(X) = \mathcal{N}(F)$. Therefore, $X = M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$.

(b) From the proof of (i)⇒ (ii), we can see that $X = M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = EUF$ for $U \in \mathbb{C}(\mathbf{z})^{k\times l}$. Next from the proof of (vi)⇒(i), if $E = ME(FME)^{(1)}FE$, and $F = FE(FME)^{(1)}FM$, for some (equivalently every) $(FME)^{(1)} \in (FME)\{1\}$, then $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ exists. Since the statements (ii) and (vi) are equivalent, therefore $E = MEUFE$, and $F = FEUFM$ imply $U \in (FME)\{1\}$. All these facts imply

$$M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = EUF = E(FME)^{(1)}F.$$

Hence, the proof is completed. □

The following corollary is obtained as a consequence of Theorem 1.

Corollary 1. *Let $M, G \in \mathbb{C}(\mathbf{z})^{n\times n}$.*

(a) *The subsequent statements are mutually equivalent:*
   - *(i)* $M^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)}$ *exists;*
   - *(ii)* *There exists $U \in \mathbb{C}(\mathbf{z})^{n\times n}$ satisfying $G = MGUG^2 = G^2UGM$;*
   - *(iii)* *There exist $U, V \in \mathbb{C}(\mathbf{z})^{n\times n}$ satisfying $G = MGUG^2 = G^2VGM$;*
   - *(iv)* *There exist $U, V \in \mathbb{C}(\mathbf{z})^{n\times n}$ satisfying $G = MGUG = GVGM$, and $GU = VG$;*
   - *(v)* *There exist $U, V \in \mathbb{C}(\mathbf{z})^{n\times n}$ satisfying $G = MVG^2 = G^2UM$, and $GU = VG$;*
   - *(vi)* $G = MG(GMG)^{(1)}G^2 = G^2(GMG)^{(1)}GM$, *for some (equivalently every)* $(GMG)^{(1)} \in (GMG)\{1\}$.

(b) *If an arbitrary of the statements* (i)-(vi) *is valid, then*

$$M^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)} = G(GMG)^{(1)}G = GUG,$$

for arbitrary fixed $(GMG)^{(1)} \in (GMG)\{1\}$ and an arbitrary $U \in \mathbb{C}(\mathbf{z})^{n\times n}$ such that the matrix equations $G = MGUG^2 = G^2UGM$ are solvable.

Theorem 6 in [41] and Corollary 2.5 in [43] can be used for finding $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$. In the following Theorem 2, we examine specific conditions on ranges and null spaces which provide commutativity of $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ with M.

Theorem 2. *Let $M \in \mathbb{C}(\mathbf{z})^{n\times n}$, $E \in \mathbb{C}(\mathbf{z})^{n\times k}$, and $F \in \mathbb{C}(\mathbf{z})^{l\times n}$ be such that $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ exists, i.e.,* $\operatorname{rank}(FME) = \operatorname{rank}(E) = \operatorname{rank}(F)$. *Then the following statements are equivalent:*

- *(i)* $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} \in M\{5\}$;
- *(ii)* $E = MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}E$, *and* $F = FM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M$;
- *(iii)* $\mathcal{R}(E) \subseteq \mathcal{R}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)})$, *and* $\mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M) \subseteq \mathcal{N}(F)$;
- *(iv)* $\mathcal{R}(E) = \mathcal{R}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)})$, *and* $\mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M) = \mathcal{N}(F)$;
- *(v)* $\operatorname{rank}(E) = \operatorname{rank}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)})$, *and* $\operatorname{rank}(F) = \operatorname{rank}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M)$;
- *(vi)* $\mathcal{R}(E) \subseteq \mathcal{R}(ME)$, *and* $\mathcal{N}(FM) \subseteq \mathcal{N}(F)$.

Proof. (i) ⇒ (ii): Using $E = M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}ME$ and $F = FMM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ in conjunction with (i), we obtain (ii).

(ii) ⇒ (iii): This implication is obvious.

(iii) ⇒ (i): Using $\mathcal{R}(E) \subseteq \mathcal{R}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)})$ and $\mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M) \subseteq \mathcal{N}(F)$, we obtain $E = MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}E$ as well $F = FM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M$. Since $\mathcal{R}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{R}(E)$ and $\mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{N}(F)$, then $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = EU = VF$, for some $U \in \mathbb{C}(\mathbf{z})^{k\times n}$ and $V \in \mathbb{C}(\mathbf{z})^{n\times l}$. Therefore,

$$M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = EU = \left(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}E\right)U = MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}(EU) = MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)},$$

and similarly $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M$. Thus

$$\begin{aligned}MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} &= M\left(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M\right) = \left(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}\right)M \\ &= M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M.\end{aligned}$$

(i) ⇔ (iv): Since $MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ and $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M$ are idempotents, then

$$\begin{aligned}&MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M \\ \Leftrightarrow\ &\mathcal{R}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{R}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M) \text{ and } \mathcal{N}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M) \\ \Leftrightarrow\ &\mathcal{R}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{R}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) \text{ and } \mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M) \\ \Leftrightarrow\ &\mathcal{R}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{R}(E) \text{ and } \mathcal{N}(F) = \mathcal{N}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M).\end{aligned}$$

(iv) ⇔ (v): This implication is obvious.

(iv) ⇒ (vi): Using $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(FME)^{(1)}F$ from ([41], [Theorem 6]) or ([43], [Corollary 2.5]), this part is evident.

(vi) ⇒ (ii): Based on $\mathcal{R}(M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}) = \mathcal{R}(E)$ it is noticed from ([41], [Theorem 3]) and ([43], [Corollary 2.1]) that

$$M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(ME)^{(1)}, \tag{2}$$

for some $(ME)^{(1)} \in (ME)\{1\}$. Similarly, from Theorem 4 in [41] and Corollary 2.3 in [43],

$$M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = (FM)^{(1)}F, \tag{3}$$

for some $(FM)^{(1)} \in (FM)\{1\}$. From Equations (2) and (3), it is concluded that $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(ME)^{(1)} = (FM)^{(1)}F$. A further consequence of the inclusion $\mathcal{R}(E) \subseteq \mathcal{R}(ME) = \mathcal{R}(ME(ME)^{(1)}) = \mathcal{N}(I - ME(ME)^{(1)})$ is

$$E = ME(ME)^{(1)}E = M(E(ME)^{(1)})E = MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}E.$$

In the same way, $\mathcal{N}(FM) \subseteq \mathcal{N}(F)$ yields $F = FM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}M$. □

The upcoming result can be verified using Theorem 2.

Corollary 2. *Let $M, G \in \mathbb{C}(\mathbf{z})^{n\times n}$ be such that $M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ exists. Then the following statements are equivalent:*

(i) $M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)} \in M\{5\}$;

(ii) $G = MM^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}G = GM^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}M$;

(iii) $\mathcal{R}(G) \subseteq \mathcal{R}(MM^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)})$, *and* $\mathcal{N}(M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}M) \subseteq \mathcal{N}(G)$;

(iv) $\mathcal{R}(G) = \mathcal{R}(MM^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)})$, *and* $\mathcal{N}(M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}M) = \mathcal{N}(G)$;

(v) $\operatorname{rank}(G) = \operatorname{rank}(MM^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}) = \operatorname{rank}(M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}M)$;

(vi) $\mathcal{R}(G) \subseteq \mathcal{R}(MG)$, *and* $\mathcal{N}(GM) \subseteq \mathcal{N}(G)$.

4. Commuting Outer Inverse-Based Solutions to Yang–Baxter-like Matrix Equation

Following the intention of the previous section, in this section we investigate the possibility of solving the YB-like equation using obtained results about $\{2,5\}$-inverses. Before proceeding, we require results included in Lemma 3.

Lemma 3. *Assume* $A, B \in \mathbb{C}(\mathbf{z})^{n\times n}$. *Then, for arbitrary* $Y \in \mathbb{C}(\mathbf{z})^{n\times n}$,

$$X = A^{\dagger}B + (I - A^{\dagger}A)BAB^{\dagger} + (I - A^{\dagger}A)Y\big(I - BB^{\dagger}\big) \tag{4}$$

is a solution to the YB-like matrix equation (1) *iff* B *satisfies*

$$ABA = B^2,\ AA^{\dagger}B = B,\ BAB^{\dagger}B = BA. \tag{5}$$

Proof. Due to ([33], [Lemma 2.3]), it is well-known that

$$AX = B,\ XB = BA, \tag{6}$$

is consistent if and only if (iff) B satisfies (5). Moreover, corresponding to each B satisfying (5), the general solution to (6) is equal to (4).

Furthermore, in ([33], [Lemma 3.1]), it was shown that X_0 solves (1) iff there exists an appropriate matrix B which ensures that X_0 is a solution to (6). This proves the lemma. □

Corollary 3. *Let* $A, B \in \mathbb{C}(\mathbf{z})^{n\times n}$ *and* A *be the nonsingular matrix. Then*

$$X = A^{-1}B \tag{7}$$

is a solution to (1) *iff* B *satisfies* $ABA = B^2$.

Proof. Since A is nonsingular, then, $\mathcal{N}(BB^{\dagger}) = \mathcal{N}(B) = \mathcal{N}(BA)$. Now using the fact that $BB^{\dagger}$ is idempotent, it follows that $BAB^{\dagger}B = BA$ in (5) holds. Furthermore, by $A^{\dagger} = A^{-1}$, Equation (4) reduces to $X = A^{-1}B$ and $AA^{\dagger}B = B$ in (5) is valid. This completes the proof. □

We define an infinite collection $\mathbb{C}(A)$ of complex numbers by

$$\mathbb{C}(A) = \{\alpha, \beta \in \mathbb{C} : \alpha I + \beta A \neq 0\}.$$

At this point, we prove the following existence result for the YB-like matrix equation corresponding to an arbitrary square matrix A.

Theorem 3. *Let* $A \in \mathbb{C}(\mathbf{z})^{n\times n}$ *be a given arbitrary matrix. Consider a nonzero matrix* $M = \alpha I + \beta A$, *where* $\alpha, \beta \in \mathbb{C}(A)$. *Suppose* $E \in \mathbb{C}(\mathbf{z})^{n\times k}$, *and* $F \in \mathbb{C}(\mathbf{z})^{l\times n}$ *be such that any of the following assumptions (A1)–(A2) holds:*

(A1) *One of the statements (i)–(vi) of part (a) in Theorem 1 is true;*

(A2) *One of statements (i)–(vi) of part (a) in Theorem 2 is true and* $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ *exists.*

Then $P := P_{\mathcal{R}(ME),\mathcal{N}(F)} = MM^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ *is an idempotent and commutes with* A. *Moreover, for* $B \in \big\{A^2P,\ A^2(I-P)\big\}$, *the matrix* X *defined by* (4) *(resp.* (7)*) is a solution to the singular (resp. nonsingular) YB-like Equation* (1).

Proof. Suppose the assumption (A1) holds. Then it follows from part (a)(i) of Theorem 1 that $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ exists. Using the facts $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}MM^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ and $MM^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}M$, it is clear that P is an idempotent and commutes with M and hence with A. The first statement is verified. For the given choice of B, it follows from [33] (Lemma 4.1) that the matrix B satisfies the conditions in (5). So by Lemma 3 (resp. Corollary 3), the second assertion follows. The remaining proof under the assumption (A2) is immediate. □

In the same way, we can present the following conclusion.

Corollary 4. *Let $A \in \mathbb{C}(\mathbf{z})^{n\times n}$ be arbitrary. Consider a nonzero matrix $M = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}(A)$. Suppose $G \in \mathbb{C}(\mathbf{z})^{n\times n}$ be such that any of the following assumptions (AS1)–(AS2) holds:*

(AS1) One of the statements (i)–(vi) of part (a) in Corollary 1 is true;

(AS2) One of statements (i)–(vi) of part (a) in Corollary 2 is true, provided $M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ exists.

Then $P := P_{\mathcal{R}(MG),\mathcal{N}(G)} = MM^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)}$ is an idempotent and commutes with A. Moreover, for the matrix $B \in \{A^2P,\ A^2(I-P)\}$, the matrix X obtained as in (4) (resp. (7)) is a solution to the singular resp. nonsingular YB-like matrix Equation (1).

Remark 1. *Let the matrix B be suggested by Theorem 3 or Corollary 4. Then,*

(a) X represented by (4) will be an infinite family of solutions of the singular YB-like Equation (1), since the involved matrix Y is arbitrary. In addition, the entries of X consist of $y_{i,j}$'s if Y is taken in the form $Y = [y_{i,j}]$. According to (4.1) in singular case, unevaluated symbols $y_{i,j}$'s are incorporated into the elements of the resulting matrix X.

(b) X represented by (7) will be a unique solution to the nonsingular YB-like Equation (1).

Theorem 1 provides not only criteria for the existence of $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$, but also a method for detecting such an inverse. More precisely, the problem of determining {2,5}-inverse X of M satisfying $\mathcal{R}(X) = \mathcal{R}(E)$ and $\mathcal{N}(X) = \mathcal{N}(F)$ reduces to finding a solution U to the system $E = MEUFE$ or $F = FEUFM$ under specific constraints. Then $\{2,5\}$-inverse X of M satisfying $\mathcal{R}(X) = \mathcal{R}(E)$ and $\mathcal{N}(X) = \mathcal{N}(F)$ can be obtained as $X = EUF$. On the other hand, Theorem 3 is based on the usage of $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ to spot the solutions of the equation $XAX = AXA$ for a suitable matrix $M \in \mathbb{C}(\mathbf{z})^{n\times n}$.

Remark 2. *Since $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} \in M\{2\}_{\mathcal{R}(E),\mathcal{N}(F)}$, it is clear from Corollary 2.5. in [43] that the existence of $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$ requires $\mathrm{rank}(FME) = \mathrm{rank}(E) = \mathrm{rank}(F)$. This condition will be exploited when making a selection of matrices E and F.*

Thus, we can state the Algorithm 1 for generating solutions to (1), according to the results presented in Theorems 1 and 3.

Algorithm 1 Solving the singular (resp. nonsingular) YB-like matrix Equation (1) using Theorem 1 in conjunction with Theorem 3.

Require: The matrix $A \in \mathbb{C}(\mathbf{z})^{n\times n}$.
1: Construct a matrix $M = \alpha I + \beta A$, in which $\alpha, \beta \in \mathbb{C}(A)$.
2: Choose suitable matrices $E \in \mathbb{C}(\mathbf{z})^{n\times k}$ and $F \in \mathbb{C}(\mathbf{z})^{l\times n}$ so that $\text{rank}(FME) = \text{rank}(E) = \text{rank}(F)$.
3: If the matrix equation $E = MEUFE$ is consistent with respect to the unknown $U \in \mathbb{C}(\mathbf{z})^{k\times l}$, then continue, else go to Step 2:.
4: Compute the output $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = EUF$.
5: Calculate $P := P_{\mathcal{R}(ME),\mathcal{N}(F)} = MM^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$.
6: For each $B \in \{A^2P,\ A^2(I-P)\}$, return X determined by (4) or (7), taking $Y = [y_{i,j}]_{n\times n}$ in symbolic form.
7: End.

We recall that Theorem 6 in [41] and Corollary 2.5 in [43] deliver two frameworks for computing $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(FME)^{(1)}F$, in case $\text{rank}(FME) = \text{rank}(E) = \text{rank}(F)$. The first approach is based on the direct computation of $(FME)^{(1)}$ and the second one is enabled in which $(FME)^{(1)}$ is calculated by solving a matrix equation $EUFME = E$ or $FMEUF = F$. On the other hand, Theorem 2 investigates equivalent axioms when $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ becomes $M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)}$.

These consequences in association with Theorem 3 make it possible to present the Algorithm 2 for producing solutions to (1).

Algorithm 2 Solving the singular (resp. nonsingular) YB-like matrix Equation (1) using Theorem 2 in conjunction with Theorem 3.

Require: The matrix $A \in \mathbb{C}(\mathbf{z})^{n\times n}$.
1: Construct a matrix $M = \alpha I + \beta A$, in which $\alpha, \beta \in \mathbb{C}(A)$.
2: Choose the suitable matrices: $E \in \mathbb{C}(\mathbf{z})^{n\times k}$ and $F \in \mathbb{C}(\mathbf{z})^{l\times n}$ so that $\text{rank}(FME) = \text{rank}(E) = \text{rank}(F)$.
3: Compute $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(FME)^{(1)}F$.
4: If $\text{rank}(E) = \text{rank}(F) = \text{rank}(MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)})$ then continue, otherwise, go to step 2:.
5: Calculate $P := P_{\mathcal{R}(ME),\mathcal{N}(F)} = MM^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$.
6: For each $B \in \{A^2P,\ A^2(I-P)\}$, return X determined by (4) or (7), taking $Y = [y_{i,j}]_{n\times n}$ in symbolic form.
7: End.

5. Implementation Details and Illustrative Experiments

This section aims to describe main implementation details and develop test examples to verify the practical applicability of theoretical findings discussed in the above sections. The key point in implementing Algorithm 1 is to solve $E = MEUFE$, required in Step 3:. On the other hand, the solution $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = E(FME)^{(1)}F$ based on an arbitrary inner inverse $(FME)^{(1)}$ in Step 4: of Algorithm 2 can be calculated by ([41], [Theorem 6]) or ([43], [Corollary 2.5]) using the following steps 3.1: and 3.2:
3.1: Solve the matrix equation $EUFME = E$ with respect to unknown matrix $U \in \mathbb{C}(\mathbf{z})^{k\times l}$.
3.2: Compute $M^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)} = EUF$.

So, the implementation Algorithm 2 is based on the matrix equation $EUFME = E$.

Step 3: in Algorithm 1 and Step 3: in Algorithm 2 in the constant matrix environment can be considered as the minimization problems

$$\min\ \varepsilon(t) = \begin{cases} \frac{\|MEU(t)FE-E\|_F^2}{2}, & \text{in Algorithm 1} \\ \frac{\|EU(t)FME-E\|_F^2}{2}, & \text{in Algorithm 2}, \end{cases} \tag{8}$$

where $t \geq 0$ is the time and $U(t)$ is unknown state variables matrix. The Gradient Neural Network (GNN) evolution from [41] based on the goal function (8) is defined by the following GNN dynamical flow:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \dot{U}(t) = \begin{cases} -\gamma(ME)^{\mathrm{T}}(MEU(t)FE-E)(FE)^{\mathrm{T}}, & \text{in Algorithm 1} \\ -\gamma E^{\mathrm{T}}(EU(t)FME-E)(FME)^{\mathrm{T}}, & \text{in Algorithm 2}. \end{cases}$$

The implementation of Step 3: in Algorithm 1 and Step 3: in Algorithm 2 in the general multivariate case was proposed in [43], and it is based on symbolic capabilities of programming package MATHEMATICA [46].

For a given matrix A and a suitable matrix B, the explicit formula (4) involves the computation of generalized inverses $A^{\dagger}$ and $B^{\dagger}$. It is worth mentioning that the Moore–Penrose inverse of an arbitrary matrix can be evaluated in MATHEMATICA through the built-in function `PseudoInverse`, whose implementation is based on its singular value decomposition.

Example 1. *Let us consider three-variable singular rational YB-like Equation (1), where*

$$A = \begin{bmatrix} z_1 & 0 & 0 & 1-z_1 & z_2-z_1 \\ 0 & z_3 & 0 & 1-z_3 & z_2-z_3 \\ z_1 & 0 & 0 & -z_1 & -z_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z_2 \end{bmatrix} \in \mathbb{C}(z_1, z_2, z_3)^{5\times 5}.$$

This example is based on Algorithm 1 for the singular case, where $M = A + I$. Furthermore, we take the following matrices E and F in conjunction with M:

$$E = \begin{bmatrix} 0 & 0 & 2 & z_2+1 \\ 0 & z_3+1 & 1-z_3 & z_2-z_3 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & z_2+1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here the matrices E and F are generated according to the rank conditions $\operatorname{rank}(FME) = \operatorname{rank}(E) = \operatorname{rank}(F)$ *The expression* `U = Table[Subscript[u,i,j], {i,4}, {j,4}]` *generates the 4×4 matrix $U = [u_{i,j}]$ with unassigned symbols $u_{i,j}$ as entries. Let* `vars = Flatten[U];` *then the general solution U is obtained using the MATHEMATICA command* `Solve[MEUFE==E, vars]//Simplify` *or* `Solve[FEUFM==F, vars]//Simplify` *and it is as follows:*

$$U = \begin{bmatrix} 1 & 0 & -\frac{3}{4} & \frac{-z_2^2-2z_2}{(z_2+1)^2} \\ 0 & \frac{1}{(z_3+1)^2} & \frac{(z_3-1)(z_3+3)}{4(z_3+1)^2} & \frac{-z_2^2-2z_2+z_3^2+2z_3}{(z_2+1)^2(z_3+1)^2} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{(z_2+1)^2} \end{bmatrix}.$$

Therefore it is justifiable to apply Theorem 1, which produces the output

$$M^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = EUF = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{z_2+1} \\ 0 & \frac{1}{z_3+1} & 0 & \frac{z_3-1}{2(z_3+1)} & \frac{z_3-z_2}{(z_2+1)(z_3+1)} \\ -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{z_2+1} \end{bmatrix}.$$

Further, simple calculations confirm that

$$P_{\mathcal{R}(ME),\mathcal{N}(F)} = MM^{(2,5)}_{\mathcal{R}(E),\mathcal{N}(F)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is idempotent and commutes with A. Since A is singular, X is given by (4); assuming $Y = [y_{i,j}]_{5\times 5}$, for the matrix $B = A^2P$, with $P := P_{\mathcal{R}(ME),\mathcal{N}(F)}$. Then one calculates

$$X = \begin{bmatrix} 0 & 0 & 0 & 1 & z_2 \\ 0 & z_3 & 0 & 1-z_3 & z_2-z_3 \\ \frac{1}{3}(y_{3,1}-y_{3,4}-y_{3,5}) & 0 & y_{3,3} & \frac{1}{3}(-y_{3,1}+y_{3,4}+y_{3,5}) & \frac{1}{3}(-y_{3,1}+y_{3,4}+y_{3,5}) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z_2 \end{bmatrix}.$$

Finally, it is possible to show that X is a solution to (1). It should be observed that the elements of X depend upon $y_{3,1}, y_{3,4}$ and $y_{3,5}$ and hence X represents an infinite solution to (1). This certifies Remark 1 (a). Similarly, X will be a solution when $B = A^2(I-P)$.

On the other hand, if we denote $P' := P_{\mathcal{R}(ME'),\mathcal{N}(F')}$ for

$$E' = \begin{bmatrix} z_1+1 & 0 \\ 0 & z_3+1 \\ z_1+1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F' = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix},$$

and choose $B \in \{A^2P',\ A^2(I-P')\}$, it can be verified that X given by (4) is a solution to (1).

Example 2 ([29]). *Let us suppose the one-variable nonsingular YB-like Equation (1) for*

$$A = \begin{bmatrix} \sin t + 2 & \cos\ t \\ -\cos t & \sin t + 3 \end{bmatrix} \in \mathbb{C}(t)^{2\times 2}.$$

Let $M = A$. The goal of this example is again to illustrate Algorithm 1 for the nonsingular case in the situation $E = F = G$, where

$$G = \begin{bmatrix} \frac{1}{2}(-1+f(t)) & \cos t \\ -\cos t & \frac{1}{2}(1+f(t)) \end{bmatrix}$$

with $f(t) = \sqrt{-1-2\cos 2t}$. For this choice of G, the solution $U = [u_{i,j}]_{2\times 2}$ to the system $MGUG^2 = G^2UGM = G$ can be obtained similarly as in above example. As a confirmation, it is equal to

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & \dfrac{\begin{array}{l}\sec^2 t((\cos 4t + \cos 2t(-2f(t)\sin t - 3f(t) + 11) \\ +2\sin 3t + f(t) + 6)u_{1,1} + 2\cos t((f(t)+4)\cos 2t \\ +f(t)\sin t + \sin 3t + 3f(t) + 2)(u_{1,2} - u_{2,1}) - 2f(t) + 2)\end{array}}{-4\cos 2t + 4f(t)\sin t + 10f(t) - 2} \end{bmatrix}.$$

So, with the help of Corollary 1, we deduce

$$M^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)} = GUG = \begin{bmatrix} \frac{-1+f(t)}{-1+5f(t)-2\cos 2t+2f(t)\sin t} & -\frac{2\cos t}{1-5f(t)+2\cos 2t-2f(t)\sin t} \\ \frac{2\cos t}{1-5f(t)+2\cos 2t-2f(t)\sin t} & \frac{1+f(t)}{-1+5f(t)-2\cos 2t+2f(t)\sin t} \end{bmatrix}.$$

Therefore, the matrix

$$P_{\mathcal{R}(MG),\mathcal{N}(G)} = MM^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)} = \begin{bmatrix} \frac{1}{2}\left(\frac{-1+f(t)}{f(t)}\right) & \frac{\cos t}{f(t)} \\ -\frac{\cos t}{f(t)} & \frac{1}{2}\left(\frac{1+f(t)}{f(t)}\right) \end{bmatrix}$$

is an idempotent and commutes with A. Since A is nonsingular, let X be the singleton matrix of the form (7). If $P := P_{\mathcal{R}(MG),\mathcal{N}(G)}$, then $X = AP$ when $B = A^2P$. In this situation, simplifications give

$$X = \begin{bmatrix} \frac{(f(t)-1)(\sin t+2)-2\cos^2 t}{2f(t)} & \frac{\cos t(2\sin t+f(t)+5)}{2f(t)} \\ -\frac{\cos t(2\sin t+f(t)+5)}{2f(t)} & \frac{(f(t)+1)(\sin t+3)-2\cos^2 t}{2f(t)} \end{bmatrix}.$$

It can be checked that X is a solution to the matrix Equation (1). Likewise, $X = A(I-P)$ by putting $B = A^2(I-P)$ is also a solution.

If we choose the matrix

$$G' = \begin{bmatrix} \frac{1}{2}(-1-f(t)) & \cos\ t \\ -\cos t & \frac{1}{2}(1-f(t)) \end{bmatrix},$$

we can show that X is a different solution to (1), whenever $B \in \{A^2P',\ A^2(I-P')\}$ in which $P' := P_{\mathcal{R}(MG'),\mathcal{N}(G')}$.

Example 3. *Let $n \in \{100, 200, 300, \ldots, 1000\}$. We deal with the constant YB-like matrix equations, where the complex matrix $A[n] \in \mathbb{C}(\mathbf{z})^{n\times n}$ is defined by*

$$A[n] = (a_{i,j}) = \begin{cases} -\frac{2}{n}\mathrm{i}, & \text{if} \quad i \neq j, \quad i,j = 1,\ldots,n, \\ \frac{2(n-j+1)}{n}\mathrm{i}, & \text{if} \quad i = j, \quad i,j = 1,\ldots,n, \end{cases}$$

and $\mathrm{i} = \sqrt{-1}$ is imaginary unit.

For a given n, here, it is easy to confirm that $A[n]$ is nonsingular. We implemented the Algorithm 2 in machine precision arithmetic to spot the solution to Equation (1) in $A[n]$ for the nonsingular case.

Let n be fixed. To illustrate the script, we take $M = A[n]$ corresponding to the choice $E = F = G$. In this example, G is taken as a singular matrix of index one commuting with M.

Since M is nonsingular, it is easy to verify that the declaration $\mathrm{rank}(GMG) = \mathrm{rank}(G)$ in Step 2 of the algorithm is satisfied. Consequently, by Lemma 2, $M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ exists. Notice that $\mathrm{rank}(MM^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}) = \mathrm{rank}(M^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}) = \mathrm{rank}(G)$. This observation shows that the condition in Step 4: of Algorithm 2 also holds.

Observe $P := P_{\mathcal{R}(MG),\mathcal{N}(G)} = MM^{(2,5)}_{\mathcal{R}(G),\mathcal{N}(G)}$. Then by the representation (7), the required approximation $X[n] = A[n]P$ can be located, with the choice $B = A^2[n]P$. Here we use the *expression* $est_{abs}(X[n]) = \|A[n]X[n]A[n] - X[n]A[n]X[n]\|_F$ *to measure the absolute error and to estimating the quality of* $X[n]$.

The results are displayed in Table 1. According to data involved in the table, $X[n]$ *is a nontrivial solution because* $\|X[n]\|_F$ *and* $\|A[n] - X[n]\|_F$ *are nonzero. From the value of* $est_{abs}(X[n])$*, it is straightforward that* $X[n]$ *is a reliable estimation of the solution from the point of accuracy.*

The numerical reports and evidence in this section clearly show a good agreement with the theoretical aspects of the paper.

Table 1. Frobenius-norm errors in Example 3.

n	$\|X[n]\|_F$	$\|A[n] - X[n]\|_F$	$\mathbf{est_{abs}}(X[n])$
100	16.3841	114.985	3.22656×10^{-10}
200	23.1327	230.457	3.29207×10^{-9}
300	28.316	345.928	9.57585×10^{-9}
400	32.6874	461.398	3.06278×10^{-8}
500	36.5395	576.868	5.71372×10^{-8}
600	40.0225	692.339	1.12476×10^{-7}
700	43.2258	807.809	1.9046×10^{-7}
800	46.2075	923.279	2.71205×10^{-7}
900	49.0082	1038.75	3.48133×10^{-7}
1000	51.6572	1154.22	4.3975×10^{-7}

6. Conclusions

The Yang–Baxter-like matrix equation has been widely studied and utilized in numerous fields of mathematics and physics. This research presents a valuable application of $\{2.5\}$-inverses in solving the Yang–Baxter matrix equation. In order to achieve this aim, we have described the set of $\{2,5\}$-inverses (termed commuting outer inverses) with predefined image and kernel as auxiliary results. Then the application of the given approach in developing the solution structures for constant as well as rational YB-like matrix equations is pointed out. In this way, algorithms defined on rational matrices are constructed based on the correlation between the symbolic calculation of the aforementioned inverses and derived explicit solutions of the matrix equation. The algorithm based on any proposed formula can be implemented in some computer algebra systems. Implementing those computational procedures in symbolic form, in the form of exact arithmetic and double-precision arithmetic is also a concern. Thus, this paper has greatly extended the previous work of [33].

Finally, some numerical experiments are performed to manifest the superiority of the proposed methods.

The results obtained in this paper are another confirmation that generalized inverses are closely related to solving the Yang–Baxter matrix equation. It is logical to assume that generalized inverses can be used in many new ways in solving this complex problem. In addition, it can be expected that other classes of generalized inverses are also included at the basis of solutions to YB-like equations. One exciting area for research may be solving the minimization problem (8) instead of finding the exact solution to embedded matrix equations employing a symbolic programming package. In addition, the rank optimization problem $\min_{AXA=B} \operatorname{rank}(B - XAX)$ could be solved in a number of different ways.

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