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Minimal Rank Properties of Outer Inverses with Prescribed Range and Null Space

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Abstract: The purpose of this paper is to investigate solvability of systems of constrained matrix equations in the form of constrained minimization problems. The main novelty of this paper is the unification of solutions of considered matrix equations with corresponding minimization problems. For a particular case we extend some well-known results and give several new results for the weak Drazin inverse. The main characterizations of the Drazin inverse, group inverse and Moore–Penrose inverse are obtained as consequences.

Keywords: matrix equation; generalized inverse; matrix rank

MSC: 15A09; 15A24; 15A23; 65F20

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1. Introduction

The set containing $m \times n$ matrices over the complex numbers $\mathbb{C}$ will be denoted as $\mathbb{C}^{m\times n}$. Standardly, A^*, $\mathrm{rk}(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ will represent the conjugate transpose, rank, range (column space) and kernel (null space), respectively. Furthermore, $\mathbb{C}_r^{m\times n} = \{X|\ X\in\mathbb{C}^{m\times n}, \mathrm{rk}(X)=r\}$.

Generalized inverses are very powerful tools in many branches of mathematics, technics and engineering. The most frequent application of generalized inverses is in finding solutions of many matrix equations and systems of linear equations. There are many other mathematical and technical disciplines in which generalized inverses play an important role. Some of them are estimation theory (regression), computing polar decomposition, electrical circuits (networks) theory, automatic control theory, filtering, difference equations, pattern recognition and image restoration. Since 1955, thousands of papers have been published discussing various theoretical and computational features of generalized inverses and their applications. For the sake of completeness, we surveyed definitions of generalized inverses related to our research.

For arbitrary $A \in \mathbb{C}^{m\times n}$, there is a Moore–Penrose inverse of A represented by the distinctive matrix $X \in \mathbb{C}^{n\times m}$ (denoted by $A^\dagger$) for which [1]:

$$(\mathbf{1})\ \ AXA = A, \quad (\mathbf{2})\ \ XAX = X, \quad (\mathbf{3})\ \ (AX)^* = AX, \quad (\mathbf{4})\ \ (XA)^* = XA.$$

The symbol $A\{\rho\}$ is stated for the set of all matrices that satisfy equations involved in $\rho \subseteq \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$. A ρ-inverse of A, marked with $A^{(\rho)}$, is any matrix from $A\{\rho\}$. Notice that $A\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\} = \{A^\dagger\}$.

The class consisting of outer generalized inverses ($\{2\}$-inverses) is defined for arbitrary $A \in \mathbb{C}^{m\times n}$ by

$$A\{2\} = \{X \in \mathbb{C}^{n\times m} |\ XAX = X\}. \tag{1}$$

Immediately from the definition, it can be concluded $\mathrm{rk}(A^{(2)}) \le \mathrm{rk}(A)$. Furthermore, it is known that an arbitrary $X \in A\{1,2\}$ satisfies $\mathrm{rk}(X) = \mathrm{rk}(A)$. The outer inverses have many applications in statistics [2,3], in the iterative themes for tackling nonlinear Equations [4], in stable approximations of ill-posed problems and in linear and nonlinear issues implicating rank-deficient generalized inverses [5].

Consider $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$. An outer inverse of A with predefined range $\mathcal{R}(B)$ (denoted by $A^{(2)}_{\mathcal{R}(B),*}$) is a solution to the following constrained equation:

$$XAX = X, \qquad \mathcal{R}(X) = \mathcal{R}(B). \tag{2}$$

The class of outer inverses with the predefined range $\mathcal{R}(B)$ is denoted by $A\{2\}_{\mathcal{R}(B),*}$. Furthermore, an outer inverse of A with given kernel $\mathcal{N}(C)$ (denoted by $A^{(2)}_{*,\mathcal{N}(C)}$) is a solution to the following constrained equation:

$$XAX = X, \qquad \mathcal{N}(X) = \mathcal{N}(C). \tag{3}$$

The symbol $A\{2\}_{*,\mathcal{N}(C)}$ will stand for the class of outer inverses with the predefined kernel $\mathcal{N}(C)$. Finally, an outer inverse of A with given image $\mathcal{R}(B)$ and kernel $\mathcal{N}(C)$ (denoted by $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$) is the unique solution of the constrained equation

$$XAX = X, \qquad \mathcal{R}(X) = \mathcal{R}(B), \qquad \mathcal{N}(X) = \mathcal{N}(C). \tag{4}$$

The key characterizations, representations and computational procedures for outer inverses with prescribed range and/or kernel were discovered in [6–10] and other research articles cited in these references. More details can be found in the monographs [4,11,12]. Full rank representations of outer inverses are given in [13,14]. Characterizations, representations and computational procedures based on appropriate matrix equations and ranks of involved matrices are proposed in [15–17]. Iterative computational algorithms were developed in [18–23].

Recall that

$$A^{\dagger} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}.$$

For $A \in \mathbb{C}^{n\times n}$, there exists the Drazin inverse A^{D} of A as the unique matrix $X \in \mathbb{C}^{n\times n}$ and it has the following properties:

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

where $k = \mathrm{ind}(A)$ is used with meaning of the index of A. That is, k is the smallest nonnegative integer satisfying $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$. Under the limitation $\mathrm{ind}(A) = 1$, the group inverse of A is $A^{\mathrm{D}} = A^{\#}$. Notice that

$$A^{\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} \qquad \text{and} \qquad A^{\#} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

The Drazin inverse proved to be useful in the investigation of finite Markov chains, in the analysis of singular linear difference equations and differential Equations [24], cryptography [25] and other.

It is important to mention that some of popular generalized inverses are outer inverses with a predefined image and kernel. One of the most popular is the core-EP inverse applicable on square matrices in [26]. For a square matrix A of index $k = \mathrm{ind}(A)$, its CEP inverse is the uniquely defined by

$$A^{\oplus}AA^{\oplus} = A^{\oplus}, \;\; \mathcal{R}(A^{\oplus}) = \mathcal{R}(A^{\oplus *}) = \mathcal{R}(A^k).$$

In the case $\mathrm{ind}(A) = 1$, the core-EP inverse transforms into the core inverse $A^{\textcircled{\#}}$ [27]. The DMP inverse $A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^{\dagger}$ is defined in [28] as the unique outer inverse satisfying $A^kX = A^kA^{\dagger}$ and $XA = A^{\mathrm{D}}A$. For arbitrary positive integer m, the m-weak group inverse (m-WGI) of a square matrix A is defined the unique solution to $AX = (A^{\oplus})^mA^m$ and

$AX^2 = X$ [29] and it can be given by $A^{\circledS m} = (A^{\oplus})^{m+1}A^m$. For $m = 1$, the m-WGI becomes the weak group inverse, proposed in [30]. For $m = 2$, the m-WGI reduces to the generalized group inverse, proposed in [31].

The definition of the weak Drazin inverse was presented in [32] as a weakened form of the Drazin inverse. Although a weak Drazin inverse lacks some properties of the Drazin inverse, such as being unique, it is still easier to find the weak Drazin inverse than the Drazin inverse. Furthermore, the weak Drazin inverse may be applied instead of the Drazin inverse; for example, in investigating differential equations or Markov chains, as well as in its additional own applications.

Consider a square matrix $A \in \mathbb{C}^{n\times n}$ of index $k = \mathrm{ind}(A)$. Then, a matrix $X \in \mathbb{C}^{n\times n}$ represents [32]

- A weak Drazin inverse of A when
$$XA^{k+1} = A^k;$$
- A minimal rank weak Drazin inverse of A when
$$XA^{k+1} = A^k \quad \text{and} \quad \mathrm{rk}(X) = \mathrm{rk}(A^{\mathrm{D}});$$
- A commuting weak Drazin inverse of A when
$$XA^{k+1} = A^k \quad \text{and} \quad AX = XA.$$

Recall that, by [32], the Drazin inverse is unique minimal rank commuting weak Drazin inverse. Important characterizations of the minimal rank weak Drazin inverse were given in [33]. Furthermore, it was proven in [33] that many recently defined generalized inverses are special cases of the minimal rank weak Drazin inverse.

The conditions for solvability of matrix equations and studying their explicit solutions were applied in physics, mechanics, control theory and many different areas [4,11]. Motivated by theoretical and applied importance of studies involving the solvability of systems of equations and forms of their solutions, we continue to study this topic.

The aim of this paper is to investigate the solvability of systems of matrix equations which are weaker than systems considered in [32,33], and to solve some constrained minimization problems. The main novelty of this paper is the unification of solutions of considered matrix equations with corresponding minimization problems. Consequently, we extend some well-known results and provide several new results for the weak Drazin inverse. Furthermore, some characterizations for significant Drazin inverse, group inverse and Moore–Penrose inverse are obtained as consequences.

2. Motivation and Research Highlights

The detailed explanations of our research goals follow in this section.

(1) For $X \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$, the first problem we consider is to find equivalent conditions for solvability of the constrained system
$$XAB = B \quad \text{and} \quad \mathrm{rk}(X) = \mathrm{rk}(B). \tag{5}$$
We will prove that X is a solution to (5) if and only if (iff) $X \in A\{2\}_{\mathcal{R}(B),*}$.

(2) In the case that system (5) is consistent, we solve the minimization model
$$\min\ \mathrm{rk}(X) \quad \text{subject to} \quad XAB = B. \tag{6}$$

(3) We investigate solvability of system (5) with the additional assumptions. Precisely, we add an additional constraint $\mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A)$ or $BAX = B$ or $AX = XA$. A minimal rank outer inverse X with prescribed range $\mathcal{R}(B)$ which commutes with A, will be called a commuting minimal rank outer inverse with prescribed range $\mathcal{R}(B)$.

(4) Suppose that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{l\times m}$. We study the solvability of the system

$$CAX = C \quad \text{and} \quad \mathrm{rk}(X) = \mathrm{rk}(C). \tag{7}$$

Since we will show that X is a solution to (7) iff $X \in A\{2\}_{*,\mathcal{N}(C)}$, a solution X to (7) is called a minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$.

(5) If the system (7) is consistent, the minimization problem

$$\min\ \mathrm{rk}(X) \quad \text{subject to} \quad CAX = C \tag{8}$$

can be solved.

(6) Special cases of the system (7) will be the topic of this research. A minimal rank outer inverse X with prescribed kernel $\mathcal{N}(C)$ which commutes with A, will be called a commuting minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$.

(7) Characterizations for the Drazin inverse, group and the Moore–Penrose inverse are obtained applying our results.

(8) The solvability of the system which contains equalities from both systems (5) and (7) is considered. Precisely, in the case that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$, we study the system

$$XAB = B, \quad CAX = C \quad \text{and} \quad \mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(C). \tag{9}$$

We will observe that X is a solution to (9) iff $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, and a solution X to (9) is called a minimal rank outer inverse with predefined range $\mathcal{R}(B)$ and kernel $\mathcal{N}(C)$. Furthermore, we investigate solvability of the system (9) with additional conditions.

The following is the organization of this paper. Preliminary information and motivation of our research are presented in Section 2. Section 3 contains investigations related to solvability of the system (5) and the minimization problem (6) as well as solvability of special cases of the system (5). As consequences, we also present characterizations for the Drazin inverse, group and the Moore–Penrose inverse. The system (7) and the minimization problem (8) are considered in Section 4. Section 5 involves solvability of the system (9) and its particular cases. Concluding remarks are part of Section 6.

3. Minimal Rank Outer Inverses with Prescribed Range

The main goals of this section are to consider solvability of the system (5) and the minimization problem (6). In the first theorem, we will observe that X presents a solution to (5) iff X is an outer inverse of A with the predefined range $\mathcal{R}(B)$. Furthermore, we give some systems of matrix equations which are equivalent to (5).

Lemma 1. **(a)** *If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$, it follows*

$$\text{there exists } X \in \mathbb{C}^{n\times m} \text{ such that } XAB = B \iff \mathrm{rk}(AB) = \mathrm{rk}(B). \tag{10}$$

(b) *For $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{l\times m}$, it follows*

$$\text{there exists } X \in \mathbb{C}^{n\times m} \text{ such that } CAX = C \iff \mathrm{rk}(CA) = \mathrm{rk}(C). \tag{11}$$

Proof. **(a)** The equality $XAB = B$ gives $\mathrm{rk}(B) \le \mathrm{rk}(AB) \le \mathrm{rk}(B)$, i.e., $\mathrm{rk}(B) = \mathrm{rk}(AB)$.

On the other hand, $\mathrm{rk}(B) = \mathrm{rk}(AB) \iff B(AB)^{(1)}AB = B$ (see, for example [11] (p. 33)), implies $XAB = B$ in the case $X = B(AB)^{(1)}$.

(b) This statement can be verified using the conjugate transpose matrices in part **(a)**. □

Theorem 1. *Suppose that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times k}$.*

(a) *The subsequent statements are mutually equivalent:*

(i) $XAB = B$ *and* $\mathrm{rk}(X) = \mathrm{rk}(B)$;

(ii) $XAB = B$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$;
(iii) X *is a solution to* (2), *i.e.*, $X \in A\{2\}_{\mathcal{R}(B),*}$;
(iv) $X = BB^{\dagger}X$ *and* $XAB = B$;
(v) $XAX = X$, $X = BB^{\dagger}X$ *and* $XAB = B$.

(b) *Additionally,*

$$\begin{aligned} &\min\{\mathrm{rk}(X)|\ XAB = B\} = \mathrm{rk}(B) \\ &\{\mathrm{rk}(X)|\ XAB = B\} \subseteq [\mathrm{rk}(B), \mathrm{rk}(X)] \\ &\{\mathrm{rk}(X)|\ X \in A\{2\} \wedge XAB = B\} \subseteq [\mathrm{rk}(B), \mathrm{rk}(A)] \end{aligned} \tag{12}$$

and the following set identities are valid:

$$A\{2\}_{\mathcal{R}(B),*} = \left\{X \in \mathbb{C}^{n\times m}|\ XAB = B\ \wedge\ \mathrm{rk}(X) = \mathrm{rk}(B)\right\} \tag{13}$$

$$A\{2\}_{\mathcal{R}(B),*} = \left\{X := B(AB)^{\dagger} + Y(I - (AB)(AB)^{\dagger})|Y \in \mathbb{C}^{n\times m} \wedge\ XAB = B \wedge \mathrm{rk}(X) = \mathrm{rk}(B)\right\}. \tag{14}$$

Proof. **(a)** (i) $\Rightarrow$ (ii): From $XAB = B$, it follows $\mathcal{R}(B) \subseteq \mathcal{R}(X)$. Furthermore, $\mathrm{rk}(X) = \mathrm{rk}(B)$ gives $\mathcal{R}(X) = \mathcal{R}(B)$.

(ii) $\Rightarrow$ (iii): The assumption $\mathcal{R}(X) = \mathcal{R}(B)$ implies $X = BW_1$ for some $W_1 \in \mathbb{C}^{k\times m}$. Then $XAX = XABW_1 = BW_1 = X$.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): It follows by (Theorem 2.3 [34]).

(v) $\Rightarrow$ (i): From $X = BB^{\dagger}X$ and $XAB = B$, it follows $\mathrm{rk}(X) = \mathrm{rk}(B)$. Furthermore, $XAB = BB^{\dagger}XAB = BB^{\dagger}B = B$.

(b) It is straightforward that $XAX = X$ implies $\mathrm{rk}(X) \leq \mathrm{rk}(A)$. On the other hand, $XAB = B$ implies $\mathrm{rk}(X) \geq \mathrm{rk}(B)$. So, (12) holds.

The set identity (13) follows from (i) $\Longleftrightarrow$ (iii). Finally, the set identities (14) follow from the general solution to the matrix equation $XAB = B$ [4,12] and the conditions (i)–(v). □

Remark that the suppositions $X = BB^{\dagger}X$ and $XAB = B$, exploited in Theorem 1, can be substituted by some of equivalent requirements presented in (Corollary 2.4 [34]).

Proposition 1. *If* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times k}$, *it follows*

there exists $X \in \mathbb{C}^{n\times m}$ *satisfying* $XAB = B$ *and* $\mathrm{rk}(X) = \mathrm{rk}(B) \Longleftrightarrow \mathrm{rk}(AB) = \mathrm{rk}(B)$.

Proof. If there exists X satisfying $XAB = B$ and $\mathrm{rk}(X) = \mathrm{rk}(B)$, by Lemma 1, we conclude $\mathrm{rk}(AB) = \mathrm{rk}(B)$.

In addition, the assumption $\mathrm{rk}(AB) = \mathrm{rk}(B)$ and (Theorem 3 [15]) imply the existence of $X \in A\{2\}_{\mathcal{R}(B),*}$. By Theorem 1, it follows $XAB = B$ and $\mathrm{rk}(X) = \mathrm{rk}(B)$. □

Because of (12), a solution X to (5) is called a minimal rank outer inverse with prescribed range $\mathcal{R}(B)$. Note that a weak Drazin inverse is a specific solution to (5) for $m = n$, $B = A^k$ and $k = \mathrm{ind}(A)$. So, we study solvability of a more general system than the system whose solution is the weak Drazin inverse.

For the particular settings $B = A^k$, $k = \mathrm{ind}(A)$ in Theorem 1, we obtain the next result which involves characterizations of the minimal rank weak Drazin inverse.

Corollary 1 generalizes results from [33], since the statements (i)–(iii) of Corollary 1 are proposed in [33].

Corollary 1. *For* $A, X \in \mathbb{C}^{n\times n}$ *and* $k \in \mathbb{N}$, *the next assertions are equivalent:*
(i) $XA^{k+1} = A^k$ *and* $\mathrm{rk}(X) = \mathrm{rk}(A^k)$;
(ii) $XA^{k+1} = A^k$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$;
(iii) $X \in A\{2\}_{\mathcal{R}(A^k),*}$;

(iv) $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$;
(v) $XAX = X$, $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$;
(vi) X *is a minimal rank weak Drazin inverse of* A.

The assumption $\mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A)$ in the system (5) reduces the results of Theorem 1 to the smaller class of inner reflexive inverses if $A\{1,2\}_{\mathcal{R}(B),*}$.

Theorem 2. *Suppose that* $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{n\times k}$.
(a) *The subsequent statements are mutually equivalent:*
(i) $XAB = B$ *and* $\mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A)$;
(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{R}(AB) = \mathcal{R}(A)$;
(iii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{R}(AB) \supseteq \mathcal{R}(A)$;
(iv) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $A = AB(AB)^\dagger A$;
(v) $XAX = X$, $AXA = A$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$, *i.e.,* $X \in A\{1,2\}_{\mathcal{R}(B),*}$.
(b) *In addition,*

$$\{X \in \mathbb{C}^{n\times m} \mid XAB = B,\ \mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A)\} = A\{1,2\}_{\mathcal{R}(B),*}. \tag{15}$$

Proof. **(a)** (i) $\Rightarrow$ (ii): According to Theorem 1, $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(B)$. Using Theorem 3, [15], $\mathrm{rk}(AB) = \mathrm{rk}(B) = \mathrm{rk}(A)$. Therefore, the fact $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ gives $\mathcal{R}(AB) = \mathcal{R}(A)$.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): These equivalences are clear.

(ii) $\Rightarrow$ (v): It is clear, by Theorem 1, that $XAB = B$. For some $V \in \mathbb{C}^{k\times n}$, the assumption $\mathcal{R}(AB) = \mathcal{R}(A)$ implies

$$A = ABV = AX(ABV) = AXA.$$

(v) $\Rightarrow$ (i): From the equalities $XAX = X$ and $AXA = A$, we deduce that $\mathrm{rk}(X) = \mathrm{rk}(A)$. The hypothesis $\mathcal{R}(X) = \mathcal{R}(B)$ yields $\mathrm{rk}(X) = \mathrm{rk}(B)$ and

$$B = XT = XA(XT) = XAB,$$

for some $T \in \mathbb{C}^{m\times k}$.

The proof of part **(b)** follows from the results of part **(a)** of this theorem. The matrices X satisfying $XAB = B$, $\mathrm{rk}(X) = \mathrm{rk}(B)$ are outer inverses of rank $\mathrm{rk}(X) = \mathrm{rk}(B) \le \mathrm{rk}(A)$. In the case $\mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A)$, outer inverses become $\{1,2\}$-inverses [15]. Consequently, the matrices X satisfying (15) are $\{1,2\}$-inverses of rank $\mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A)$. □

Proposition 2. *If* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times k}$, *it follows*

there exists $X \in \mathbb{C}^{n\times m}$ *that fulfills*
$$XAB = B \text{ and } \mathrm{rk}(X) = \mathrm{rk}(B) = \mathrm{rk}(A) \Longleftrightarrow \mathrm{rk}(AB) = \mathrm{rk}(B) = \mathrm{rk}(A).$$

When we add the assumption $AX = XA$ in the system (5), we obtain the following characterizations for a commuting minimal rank outer inverse with prescribed range $\mathcal{R}(B)$.

Theorem 3. *For* $A, X, B \in \mathbb{C}^{n\times n}$, *the subsequent statements are equivalent each other:*
(i) $XAB = B$, $\mathrm{rk}(X) = \mathrm{rk}(B)$ *and* $AX = XA$;
(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $AX = XA$;
(iii) $X^2A = AX^2 = X$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$;
(iv) $X^2A = AX^2 = X$, $X = BB^\dagger X$ *and* $XAB = B$.

Proof. (i) $\Leftrightarrow$ (ii): It follows by Theorem 1.

(ii) $\Rightarrow$ (iii): This implication is evident.

(iii) $\Rightarrow$ (ii): Using $X^2A = AX^2 = X$, we get $AX = AX^2A = XA$. Hence, $X = X^2A = XAX$.

(iv) $\Leftrightarrow$ (iii): Applying Theorem 1, one can verify this implication. $\square$

By Theorem 3, we get the next consequence which contains several characterizations for the Drazin inverse. For $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{ind}(A)$, recall that by (Corollary 2.3 [33]), X is a minimal rank weak Drazin inverse of A and $AX = XA$ iff $X = A^{\mathrm{D}}$.

Corollary 2. *Let $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$. The subsequent statements are equivalent each other:*

(i) $XA^{k+1} = A^k$, $\mathrm{rk}(X) = \mathrm{rk}(A^k)$ *and* $AX = XA$;
(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $AX = XA$;
(iii) $X^2A = AX^2 = X$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$;
(iv) $X^2A = AX^2 = X$, $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$;
(v) $X = A^{\mathrm{D}}$.

In the case that the hypothesis $BAX = B$ is added to the system (5), we present necessary and sufficient requirements for the solvability of novel system. The system $XAB = BAX = B$ was considered in [35], but in conjunction with additional assumptions different from our conditions in Theorem 4.

Theorem 4. *The subsequent statements are equivalent each other for $A, X, B \in \mathbb{C}^{n\times n}$:*

(i) $XAB = BAX = B$ *and* $\mathrm{rk}(X) = \mathrm{rk}(B)$;
(ii) $XAB = B$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(B)$;
(iii) $XAB = B$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(B)$;
(iv) $XAB = B$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(X)$;
(v) $XAB = B$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(X)$;
(vi) $XAX = X$, $BAX = B$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$;
(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(B)$, *i.e.*, $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(B)}$;
(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(B)$;
(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(X)$.

Proof. (i) $\Rightarrow$ (ii): Firstly, $BAX = B$ gives $\mathcal{N}(X) \subseteq \mathcal{N}(B)$. Since $\mathrm{rk}(X) = \mathrm{rk}(B)$, then $\dim \mathcal{N}(X) = n - \mathrm{rk}(X) = n - \mathrm{rk}(B) = \dim \mathcal{N}(B)$. So, $\mathcal{N}(X) = \mathcal{N}(B)$.

(ii) $\Rightarrow$ (iii) and (iv): It is evident.

(iii) $\Rightarrow$ (i): Theorem 1 and assumptions $XAB = B$ and $\mathcal{R}(X) = \mathcal{R}(B)$ imply $XAX = X$ and $\mathrm{rk}(X) = \mathrm{rk}(B)$. The condition $\mathcal{N}(X) \subseteq \mathcal{N}(B)$ yields, for some $V \in \mathbb{C}^{n\times n}$,

$$B = VX = (VX)AX = BAX.$$

(iv) $\Rightarrow$ (v): This implication is evident.

(v) $\Rightarrow$ (ii): From $XAB = B$, we conclude that $\mathcal{R}(B) \subseteq \mathcal{R}(X)$ and $\mathrm{rk}(B) \leq \mathrm{rk}(X)$. Because $\mathcal{N}(B) \subseteq \mathcal{N}(X)$, we have $X = SB$, for some $S \in \mathbb{C}^{n\times n}$, and so $\mathrm{rk}(X) \leq \mathrm{rk}(B)$. Hence, $\mathrm{rk}(X) = \mathrm{rk}(B)$, which implies $\mathcal{N}(X) = \mathcal{N}(B)$ and $\mathcal{R}(B) = \mathcal{R}(X)$.

The rest follows by Theorem 1. $\square$

As a consequence of Theorem 4, we get the following result which involves characterizations of the Drazin inverse.

Corollary 3. *Let $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$. The subsequent statements are mutually equivalent:*

(i) $XA^{k+1} = A^{k+1}X = A^k$ *and* $\mathrm{rk}(X) = \mathrm{rk}(A^k)$;
(ii) $XA^{k+1} = A^k$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;
(iii) $XA^{k+1} = A^k$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$;
(iv) $XA^{k+1} = A^k$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$;
(v) $XA^{k+1} = A^k$ *and* $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$;

(vi) $XAX = X$, $A^{k+1}X = A^k$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$;
(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$, $\mathcal{N}(X) = \mathcal{N}(A^k)$, *i.e.*, $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} = A^{\mathrm{D}}$;
(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$;
(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$.

For $k = 1$ in Corollary 3, we obtain characterizations for the group inverse.

Corollary 4. *The subsequent statements are equivalent for* $A, X \in \mathbb{C}^{n\times n}$*:*
(i) $XA^2 = A^2X = A$ *and* $\mathrm{rk}(X) = \mathrm{rk}(A)$;
(ii) $XA^2 = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) = \mathcal{N}(A)$;
(iii) $XA^2 = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A)$;
(iv) $XA^2 = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(A) \subseteq \mathcal{N}(X)$;
(v) $XA^2 = A$ *and* $\mathcal{N}(A) \subseteq \mathcal{N}(X)$;
(vi) $XAX = X$, $A^2X = A$ *and* $\mathcal{R}(X) = \mathcal{R}(A)$;
(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A)$, $\mathcal{N}(X) = \mathcal{N}(A)$, *i.e.*, $X = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)} = A^{\#}$;
(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A)$;
(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(A) \subseteq \mathcal{N}(X)$.

Theorem 4 also implies new characterizations for the Moore–Penrose inverse.

Corollary 5. *The next assertions are mutually equivalent for* $A, X \in \mathbb{C}^{n\times n}$*:*
(i) $XAA^* = A^*AX = A^*$ *and* $\mathrm{rk}(X) = \mathrm{rk}(A^*)$;
(ii) $XAA^* = A^*$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$;
(iii) $XAA^* = A^*$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^*)$;
(iv) $XAA^* = A^*$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$;
(v) $XAA^* = A^*$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$;
(vi) $XAX = X$, $A^*AX = A^*$ *and* $\mathcal{R}(X) = \mathcal{R}(A^*)$;
(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$, *i.e.*,
$X = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)} = A^{\dagger}$;
(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^*)$;
(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$.

Example 1. *Consider the matrices*

$$A = \begin{bmatrix} \epsilon+1 & \epsilon & \epsilon & \epsilon & \epsilon+1 \\ \epsilon & \epsilon-1 & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon+1 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon-1 & \epsilon \\ \epsilon+1 & \epsilon & \epsilon & \epsilon & \epsilon+1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2\epsilon+1 & \epsilon & \epsilon \\ \epsilon & 2\epsilon-1 & \epsilon \\ \epsilon & \epsilon & 2\epsilon+1 \\ \epsilon & \epsilon & \epsilon \\ 3\epsilon & \epsilon & \epsilon \end{bmatrix}.$$

Let us generate the candidate solutions X in the generic form

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix}, \tag{16}$$

where $x_{i,j}$, $i,j = 1,\ldots,5$ are unevaluated symbols. The general solution X to $XAB = B$ is the matrix

$$\left[\begin{matrix} x_{1,1} & \frac{2\epsilon^3+\epsilon^2-2\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{1,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-2\epsilon+(6\epsilon+3)x_{1,1}+(6\epsilon+3)x_{1,5}-3}{6\epsilon+4} \\ x_{2,1} & \frac{-7\epsilon^3+3\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{2,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{2,5}}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{\epsilon+(6\epsilon+3)x_{2,1}+(6\epsilon+3)x_{2,5}}{6\epsilon+4} \\ x_{3,1} & \frac{\epsilon(\epsilon+1)^2+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{3,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{3,5}}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{5\epsilon+(6\epsilon+3)x_{3,1}+(6\epsilon+3)x_{3,5}+4}{6\epsilon+4} \\ x_{4,1} & \frac{-\epsilon(\epsilon+1)^2+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{4,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{4,5}}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{\epsilon+(6\epsilon+3)x_{4,1}+(6\epsilon+3)x_{4,5}}{6\epsilon+4} \\ x_{5,1} & \frac{\epsilon\left(5\epsilon^2-2\epsilon-3\right)+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{5,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{5,5}}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-5\epsilon+(6\epsilon+3)x_{5,1}+(6\epsilon+3)x_{5,5}}{6\epsilon+4} \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{4\epsilon^3-\epsilon^2-2\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{1,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{1,5} \\ \frac{\epsilon\left(12\epsilon^3+3\epsilon^2-6\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{2,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{2,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{2,5} \\ \frac{-12\epsilon^4-3\epsilon^3+6\epsilon^2+\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{3,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{3,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{3,5} \\ \frac{\epsilon\left(7\epsilon^2+2\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{4,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{4,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{4,5} \\ \frac{\epsilon\left(\epsilon^2+2\epsilon-3\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{5,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{5,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{5,5} \end{matrix}\right]$$

which satisfies $XAB = B$ but does not satisfy $XAX = X$. Ranks of relevant matrices are equal to

$$\mathrm{rk}(B) = \mathrm{rk}(AB) = 3 < \mathrm{rk}(A) = 4 < \mathrm{rk}(X) = 5.$$

The matrix Z obtained by the replacement $x_{1,5} = x_{2,5} = x_{3,5} = x_{4,5} = x_{5,5} = 0$ in X is equal to

$$Z = \begin{bmatrix} 0 & \frac{2\epsilon^3+\epsilon^2-2\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-2\epsilon+(6\epsilon+3)x_{1,5}-3}{6\epsilon+4} & \frac{4\epsilon^3-\epsilon^2-2\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{1,5} \\ 0 & \frac{-7\epsilon^3+3\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{2,5}}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{\epsilon+(6\epsilon+3)x_{2,5}}{6\epsilon+4} & \frac{\epsilon\left(12\epsilon^3+3\epsilon^2-6\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{2,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{2,5} \\ 0 & \frac{\epsilon(\epsilon+1)^2+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{3,5}}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{5\epsilon+(6\epsilon+3)x_{3,5}+4}{6\epsilon+4} & \frac{-12\epsilon^4-3\epsilon^3+6\epsilon^2+\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{3,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{3,5} \\ 0 & \frac{\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{4,5}-\epsilon(\epsilon+1)^2}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{\epsilon+(6\epsilon+3)x_{4,5}}{6\epsilon+4} & \frac{\epsilon\left(7\epsilon^2+2\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{4,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{4,5} \\ 0 & \frac{\epsilon\left(5\epsilon^2-2\epsilon-3\right)+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{5,5}}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{(6\epsilon+3)x_{5,5}-5\epsilon}{6\epsilon+4} & \frac{\epsilon\left(\epsilon^2+2\epsilon-3\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{5,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{5,5} \end{bmatrix}$$

and satisfies $\mathrm{rk}(Z) = 4 > \mathrm{rk}(B)$. Then the matrix equation $ZAB = B$ holds, but $ZAZ = Z$ does not hold.

Finally, consider the matrix Q obtained by the replacement $x_{1,5} = x_{2,5} = x_{3,5} = x_{4,5} = x_{5,5} = 0$ in the matrix Z:

$$Q = \begin{bmatrix} 0 & \frac{2\epsilon^3+\epsilon^2-2\epsilon-1}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-2\epsilon-3}{6\epsilon+4} & \frac{4\epsilon^3-\epsilon^2-2\epsilon-1}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & 0 \\ 0 & \frac{3\epsilon-7\epsilon^3}{2\epsilon(3\epsilon^2-\epsilon-2)} & \frac{\epsilon}{6\epsilon+4} & \frac{12\epsilon^3+3\epsilon^2-6\epsilon-1}{4(\epsilon-1)\epsilon(3\epsilon+2)} & 0 \\ 0 & \frac{(\epsilon+1)^2}{2(3\epsilon^2-\epsilon-2)} & \frac{5\epsilon+4}{6\epsilon+4} & \frac{-12\epsilon^4-3\epsilon^3+6\epsilon^2+\epsilon}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & 0 \\ 0 & -\frac{(\epsilon+1)^2}{2(3\epsilon^2-\epsilon-2)} & \frac{\epsilon}{6\epsilon+4} & \frac{7\epsilon^2+2\epsilon-1}{4(\epsilon-1)\epsilon(3\epsilon+2)} & 0 \\ 0 & \frac{5\epsilon^2-2\epsilon-3}{2(\epsilon-1)(3\epsilon+2)} & -\frac{5\epsilon}{6\epsilon+4} & \frac{\epsilon^2+2\epsilon-3}{4(\epsilon-1)\epsilon(3\epsilon+2)} & 0 \end{bmatrix}.$$

The matrix Q *satisfies* $\mathrm{rk}(Q) = 3 = \mathrm{rk}(B)$. *Then both the matrix equations* $QAB = B$ *and* $QAQ = Q$ *are satisfied, which is in accordance with the results presented in Theorem 1.*

Now, let us calculate the matrix $X = BU$, *where* $U \in \mathbb{C}^{5\times 3}$ *is in generic form*

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,5} \\ u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} & u_{2,5} \\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} & u_{3,5} \end{bmatrix}.$$

The set of solutions to $BUAB = B$ *with respect to* U *is given by*

$$\left[\begin{matrix} u_{1,1} & u_{1,2} & \frac{3\epsilon\left(\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}+1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\ u_{2,1} & u_{2,2} & -\frac{3\epsilon(2\epsilon+1)((\epsilon-1)u_{2,2}+1)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\ u_{3,1} & u_{3,2} & \frac{6\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{3,2}\epsilon-3\epsilon-1}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \end{matrix}\right.$$
$$\left.\begin{matrix} \frac{\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}-\epsilon\left(6\epsilon^2+9\epsilon+1\right)}{2\epsilon(6\epsilon^3-3\epsilon^2-6\epsilon-1)} & \frac{3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\ \frac{24\epsilon^3+26\epsilon^2+9\epsilon+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}+1}{2\epsilon(6\epsilon^3-3\epsilon^2-6\epsilon-1)} & \frac{\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)((\epsilon-1)u_{2,2}+1)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\ \frac{\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{3,2}-4\epsilon^2(4\epsilon+1)}{2\epsilon(6\epsilon^3-3\epsilon^2-6\epsilon-1)} & \frac{\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \end{matrix}\right].$$

Then the set $A\{2\}_{\mathcal{R}(B),*}$ *coincides with the set* $Y = BU$ *which is given in Appendix A.*

The rank identities $\mathrm{rk}(Y) = \mathrm{rk}(B)$ *are satisfied.*

4. Minimal Rank Outer Inverses with Prescribed Kernel

This section is devoted to the solvability of the system (7) as well as the minimization problem (8). Besides some systems of matrix equations which are equivalent to the system (7), we present in Theorem 5 that X is a solution to the system (7) iff X is an outer inverse of A with the given kernel $\mathcal{N}(C)$.

Theorem 5. *Let* $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ *and* $C \in \mathbb{C}^{l\times m}$.

(a) *The subsequent statements are mutually equivalent:*

(i) $CAX = C$ *and* $\mathrm{rk}(X) = \mathrm{rk}(C)$;

(ii) $CAX = C$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;

(iii) X *is a solution to* (3), *i.e.,* $X \in A\{2\}_{*,\mathcal{N}(C)}$;

(iv) $X = XC^\dagger C$ *and* $CAX = C$;

(v) $XAX = X$, $X = XC^\dagger C$ *and* $CAX = C$.

(b) *In addition,*

$$\begin{aligned} &\min\{\mathrm{rk}(X)|\ CAX = C\} = \mathrm{rk}(C) \\ &\{\mathrm{rk}(X)|\ CAX = C\} \subseteq [\mathrm{rk}(C), \mathrm{rk}(X)] \\ &\{\mathrm{rk}(X)|\ X \in A\{2\} \wedge CAX = C\} \subseteq [\mathrm{rk}(C), \mathrm{rk}(A)] \end{aligned} \tag{17}$$

and the following set identities are valid:

$$A\{2\}_{*,\mathcal{N}(C)} = \left\{X \in \mathbb{C}^{n\times m}|\ CAX = C\ \wedge\ \mathrm{rk}(X) = \mathrm{rk}(C)\right\}. \tag{18}$$

$$A\{2\}_{*,\mathcal{N}(C)} = \left\{X := (CA)^\dagger C + (I - (CA)^\dagger CA)Y|\ Y \in \mathbb{C}^{n\times m} \wedge\ CAX = C \wedge \mathrm{rk}(X) = \mathrm{rk}(C)\right\}. \tag{19}$$

Proof. (i) $\Rightarrow$ (ii): The hypothesis $CAX = C$ implies $\mathcal{N}(X) \subseteq \mathcal{N}(C)$. Since $\mathrm{rk}(X) = \mathrm{rk}(C)$, we deduce that $\mathcal{N}(X) = \mathcal{N}(C)$.

(ii) $\Rightarrow$ (iii): From $\mathcal{N}(X) = \mathcal{N}(C)$, we haveit follows $X = W_2C$ for some $W_2 \in \mathbb{C}^{n\times l}$. Then $XAX = W_2CAX = W_2C = X$.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): These equivalences are clear by (Theorem 2.6 [34]).

(v) $\Rightarrow$ (i): The assumptions $X = XC^\dagger C$ and $CAX = C$ give $\mathrm{rk}(X) = \mathrm{rk}(C)$. Now, $CAX = CAXC^\dagger C = CC^\dagger C = C$.

The rest of the proof is analogous as the proof of Theorem 1. □

In order to provide new systems of matrix equations, we can replace the conditions $X = XC^{\dagger}C$ and $CAX = C$ of Theorem 5 with some of the equivalent conditions presented in (Remark 2.7 [34]).

Proposition 3. *If $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{l\times m}$, it follows*

$$\text{there exists } X \in \mathbb{C}^{n\times m} \text{ satisfying } CAX = C \text{ and } \mathrm{rk}(X) = \mathrm{rk}(C) \iff \mathrm{rk}(CA) = \mathrm{rk}(C).$$

Because of (17), a solution X to (7) is called a minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$.

Theorem 5 implies the following result.

Corollary 6. *The next statements are equivalent each other for $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$:*
(i) $A^{k+1}X = A^k$ *and* $\mathrm{rk}(X) = \mathrm{rk}(A^k)$;
(ii) $A^{k+1}X = A^k$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;
(iii) $X \in A\{2\}_{*,\mathcal{N}(A^k)}$;
(iv) $X = X(A^k)^{\dagger}A^k$ *and* $A^{k+1}X = A^k$;
(v) $XAX = X$, $X = X(A^k)^{\dagger}A^k$ *and* $A^{k+1}X = A^k$;
(vi) *X is a minimal rank weak Drazin inverse of A.*

We now consider the solvability of particular cases of the system (7). Firstly, we assume that $\mathrm{rk}(X) = \mathrm{rk}(C) = \mathrm{rk}(A)$ holds in the system (7). Notice that the following result can be proven as corresponding results of the previous section.

Theorem 6. *Consider $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{l\times m}$.*
(a) *The subsequent statements are mutually equivalent:*
(i) $CAX = C$ *and* $\mathrm{rk}(X) = \mathrm{rk}(C) = \mathrm{rk}(A)$;
(ii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{N}(A) = \mathcal{N}(CA)$;
(iii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{N}(CA) \subseteq \mathcal{N}(A)$;
(iv) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $A = A(CA)^{\dagger}CA$;
(v) $XAX = X$, $AXA = A$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$, *i.e.,* $X \in A\{1,2\}_{*,\mathcal{N}(C)}$.
(b) *In addition,*

$$\{X \in \mathbb{C}^{n\times m} \mid CAX = C,\ \mathrm{rk}(X) = \mathrm{rk}(C) = \mathrm{rk}(A)\} = A\{1,2\}_{*,\mathcal{N}(C)}. \tag{20}$$

Proposition 4. *If $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{l\times m}$, it follows*

$$\text{there exists } X \in \mathbb{C}^{n\times m} \text{ satisfying}$$
$$CAX = C \text{ and } \mathrm{rk}(X) = \mathrm{rk}(C) = \mathrm{rk}(A) \iff \mathrm{rk}(CA) = \mathrm{rk}(C) = \mathrm{rk}(A).$$

Several characterizations of a commuting minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$ are proposed in Theorem 7.

Theorem 7. *Let $A, X, C \in \mathbb{C}^{n\times n}$. The subsequent statements are mutually equivalent:*
(i) $CAX = C$, $\mathrm{rk}(X) = \mathrm{rk}(C)$ *and* $AX = XA$;
(ii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $AX = XA$;
(iii) $X^2A = AX^2 = X$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;
(iv) $X^2A = AX^2 = X$, $X = XC^{\dagger}C$ *and* $CAX = C$.

Theorem 7 gives the next result which gives characterizations of the Drazin inverse.

Corollary 7. *The subsequent statements are equivalent for $A, X, C \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$:*

(i) $A^{k+1}X = A^k$, $\mathrm{rk}(X) = \mathrm{rk}(A^k)$ *and* $AX = XA$;

(ii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $AX = XA$;

(iii) $X^2A = AX^2 = X$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;

(iv) $X^2A = AX^2 = X$, $X = X(A^k)^{\dagger}A^k$ *and* $A^{k+1}X = A^k$;

(v) $X = A^{\mathrm{D}}$.

Taking that $XAC = C$ in the system (7), we establish some necessary and sufficient conditions for a matrix X to be a solution to a novel system.

Theorem 8. *Let $A, X, C \in \mathbb{C}^{n\times n}$. The subsequent statements are equivalent each other:*

(i) $CAX = XAC = C$ *and* $\mathrm{rk}(X) = \mathrm{rk}(C)$;

(ii) $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) = \mathcal{R}(C)$;

(iii) $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(C)$;

(iv) $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(C) \subseteq \mathcal{R}(X)$;

(v) $CAX = C$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(C)$;

(vi) $XAX = X$, $XAC = C$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;

(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) = \mathcal{R}(C)$, *i.e.*, $X = A^{(2)}_{\mathcal{R}(C),\mathcal{N}(C)}$;

(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(C)$;

(ix) $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(C) \subseteq \mathcal{R}(X)$.

Consequently, by Theorem 8, we derive the following characterizations for the Drazin inverse.

Corollary 8. *The next statements are equivalent for $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$:*

(i) $A^{k+1}X = A^k$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) = \mathcal{N}(A^k)$;

(ii) $A^{k+1}X = A^k$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$;

(iii) $A^{k+1}X = A^k$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$;

(iv) $A^{k+1}X = A^k$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$;

(v) $XAX = X$, $XA^{k+1} = A^k$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;

(vi) $XAX{=}X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$, $\mathcal{R}(X) = \mathcal{R}(A^k)$, *i.e.*, $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} = A^{\mathrm{D}}$;

(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$;

(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$.

By Corollary 8, we characterize the group inverse.

Corollary 9. *The subsequent constrained equations are equivalent for $A, X \in \mathbb{C}^{n\times n}$:*

(i) $A^2X = A$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) = \mathcal{N}(A)$;

(ii) $A^2X = A$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A)$;

(iii) $A^2X = A$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(A) \subseteq \mathcal{R}(X)$;

(iv) $A^2X = A$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A)$;

(v) $XAX = X$, $XA^2 = A$ *and* $\mathcal{N}(X) = \mathcal{N}(A)$;

(vi) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) = \mathcal{R}(A)$, *i.e.*, $X = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)} = A^{\#}$;

(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A)$;

(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(A) \subseteq \mathcal{R}(X)$.

According to Theorem 8, we have more characterizations of the Moore–Penrose inverse.

Corollary 10. *The subsequent constrained equations are equivalent for* $A, X \in \mathbb{C}^{n\times n}$*:*

(i) $A^*AX = A^*, \mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) = \mathcal{R}(A^*)$;
(ii) $A^*AX = A^*, \mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$;
(iii) $A^*AX = A^*, \mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(A^*) \subseteq \mathcal{R}(X)$;
(iv) $A^*AX = A^*$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$;
(v) $XAX = X, XAA^* = A^*$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$;
(vi) $XAX = X, \mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) = \mathcal{R}(A^*)$, *i.e.,* $X = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)} = A^\dagger$;
(vii) $XAX = X, \mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$;
(viii) $XAX = X, \mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(A^*) \subseteq \mathcal{R}(X)$.

Example 2. *Consider the matrix A from Example 1 and the matrix C of rank* 3 *defined by*

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

Let us generate the candidate solutions X in the generic form (16). The general solution X to $CAX = C$ *is equal to*

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ -\frac{2\epsilon+(9\epsilon+2)x_{3,1}}{9\epsilon-2} & \frac{-5\epsilon-(9\epsilon+2)x_{3,2}+2}{9\epsilon-2} & \frac{5\epsilon-(9\epsilon+2)x_{3,3}+2}{9\epsilon-2} & -\frac{2\epsilon+(9\epsilon+2)x_{3,1}}{9\epsilon-2}\frac{4\epsilon-(9\epsilon+2)x_{3,4}}{9\epsilon-2} & -\frac{2\epsilon+(9\epsilon+2)x_{3,5}}{9\epsilon-2} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ -\frac{3\epsilon+(9\epsilon+4)x_{3,1}}{9\epsilon-2} & \frac{6\epsilon-(9\epsilon+4)x_{3,2}}{9\epsilon-2} & \frac{3\epsilon-(9\epsilon+4)x_{3,3}+4}{9\epsilon-2} & \frac{-3\epsilon-(9\epsilon+4)x_{3,4}+2}{9\epsilon-2} & -\frac{3\epsilon+(9\epsilon+4)x_{3,5}}{9\epsilon-2} \\ \frac{5\epsilon+(2-9\epsilon)x_{1,1}+(9\epsilon-1)x_{3,1}-1}{9\epsilon-2} & \frac{-\epsilon+(2-9\epsilon)x_{1,2}+(9\epsilon-1)x_{3,2}}{9\epsilon-2} & \frac{-8\epsilon+(2-9\epsilon)x_{1,3}+(9\epsilon-1)x_{3,3}+1}{9\epsilon-2} & \frac{-\epsilon+(2-9\epsilon)x_{1,4}+(9\epsilon-1)x_{3,4}}{9\epsilon-2} & \frac{5\epsilon+(2-9\epsilon)x_{1,5}+(9\epsilon-1)x_{3,5}-1}{9\epsilon-2} \end{bmatrix}.$$

The matrix X satisfies $CAX = C$ *but does not satisfy* $XAX = X$. *Ranks of relevant matrices are equal to*

$$\mathrm{rk}(C) = \mathrm{rk}(CA) = 3 < \mathrm{rk}(A) = 4 < \mathrm{rk}(X) = 5.$$

The matrix Z obtained by the replacement $x_{1,1} = x_{1,2} = x_{1,3} = x_{1,4} = x_{1,5} = 0$ *in X satisfies* $\mathrm{rk}(Z) = 4 > \mathrm{rk}(B)$. *Then the matrix equation* $ZAB = B$ *holds, but* $ZAZ = Z$ *does not hold.*

Finally, consider the matrix Q obtained by the replacement $x_{3,1} = x_{3,2} = x_{3,3} = x_{3,4} = x_{3,5} = 0$ *in Z:*

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{2\epsilon}{9\epsilon-2} & \frac{2-5\epsilon}{9\epsilon-2} & \frac{5\epsilon+2}{9\epsilon-2} & \frac{4\epsilon}{9\epsilon-2} & -\frac{2\epsilon}{9\epsilon-2} \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{3\epsilon}{9\epsilon-2} & \frac{6\epsilon}{9\epsilon-2} & \frac{3\epsilon+4}{9\epsilon-2} & \frac{2-3\epsilon}{9\epsilon-2} & -\frac{3\epsilon}{9\epsilon-2} \\ \frac{5\epsilon-1}{9\epsilon-2} & -\frac{\epsilon}{9\epsilon-2} & \frac{1-8\epsilon}{9\epsilon-2} & -\frac{\epsilon}{9\epsilon-2} & \frac{5\epsilon-1}{9\epsilon-2} \end{bmatrix}.$$

The matrix Q satisfies $\mathrm{rk}(Q) = 3 = \mathrm{rk}(B)$. *Then both the matrix equations* $QAB = B$ *and* $QAQ = Q$ *are satisfied, which is in accordance with the results presented in Theorem 5.*

Now, let us calculate the matrix $X = UC$, *where* $U \in \mathbb{C}^{5\times 3}$ *is in generic form*

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \\ u_{4,1} & u_{4,2} & u_{4,3} \\ u_{5,1} & u_{5,2} & u_{5,3} \end{bmatrix}.$$

The set of solutions to $CAUC = C$ with respect to U is given by

$$\begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & 1-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2} & u_{2,3} \\ \frac{(2-9\epsilon)u_{2,1}-6\epsilon}{9\epsilon+2} & u_{3,2} & \frac{\epsilon+(2-9\epsilon)u_{2,3}+2}{9\epsilon+2} \\ \frac{6\epsilon+(9\epsilon+4)u_{2,1}+2}{9\epsilon+2} & -\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2} \\ \frac{-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1}{9\epsilon+2} & \frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2}-u_{1,2} & \frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2} \end{bmatrix}.$$

Then the set $A\{2\}_{*,\mathcal{N}(C)}$ coincides with the set $Y = UC$ is given in Appendix B. The rank identities $\text{rk}(Y) = \text{rk}(C)$ *are satisfied.*

5. Minimal Rank Outer Inverses with Prescribed Range and Kernel

Applying results of Sections 3 and 4, we are able to characterize solvability of the system (9). In particular, by Theorem 1 and Theorem 5, the system (9) has a solution X iff X is an outer inverse of A with the prescribed range $\mathcal{R}(B)$ and kernel $\mathcal{N}(C)$.

Corollary 11. *Consider $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$.*
(a) *The subsequent constrained matrix equations are mutually equivalent:*

(i) $XAB = B$, $CAX = C$ *and* $\text{rk}(X) = \text{rk}(B) = \text{rk}(C)$;
(ii) $XAB = B$, $CAX = C$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;
(iii) *X is a solution to (4), i.e.,* $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$;
(iv) $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$;
(v) $XAX = X$, $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$.

(b) *In addition, the system (9) has the unique solution* $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

Theorem 2 and Theorem 6 imply the next characterizations of solution to the special system of the system (9) with $\text{rk}(X) = \text{rk}(B) = \text{rk}(C) = \text{rk}(A)$.

Corollary 12. **(a)** *The subsequent constrained equations are equivalent for $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$:*

(i) $XAB = B$, $CAX = C$ *and* $\text{rk}(X) = \text{rk}(B) = \text{rk}(C) = \text{rk}(A)$;
(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$, $\mathcal{R}(A) = \mathcal{R}(AB)$ *and* $\mathcal{N}(A) = \mathcal{N}(CA)$;
(iii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$, $\mathcal{R}(A) \subseteq \mathcal{R}(AB)$ *and* $\mathcal{N}(CA) \subseteq \mathcal{N}(A)$;
(iv) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $A = AB(AB)^{\dagger}A = A(CA)^{\dagger}CA$;
(v) $XAX = X$, $AXA = A$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$, *i.e.,* $X \in A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)}$.

(b) *In addition, the constrained system in* (i) *has the unique solution* $X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

Using Theorem 3 and Theorem 7, we characterize the solvability of a new system obtained from the system (9) adding an extra condition $AX = XA$.

Corollary 13. *The subsequent constrained equations are equivalent for $A, X, B, C \in \mathbb{C}^{n\times n}$:*

(i) $XAB = B$, $CAX = C$, $\text{rk}(X) = \text{rk}(B) = \text{rk}(C)$ *and* $AX = XA$;
(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $AX = XA$;
(iii) $X^2A = AX^2 = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;
(iv) $X^2A = AX^2 = X$, $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$.

Example 3. *Consider*

$$A = \begin{bmatrix} \frac{1}{\epsilon} & \theta & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & \epsilon^3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Let us generate the possible solutions Q in the generic form

$$Q = \begin{bmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{bmatrix},$$

where $q_{i,j}$, $i, j = 1, \ldots, 3$ *are unevaluated symbols. The general solution Q to the system of matrix equations* $QAB = B, CAQ = C$ *is equal to*

$$Q = \begin{bmatrix} 0 & 0 & \epsilon - \epsilon\theta\, q_{2,3} \\ \frac{1}{\theta} & 0 & x_{2,3} \\ -\frac{1}{\theta^2} & \frac{1}{\theta} & -\frac{q_{2,3}}{\theta} \end{bmatrix}.$$

Ranks of relevant matrices are equal to

$$\mathrm{rk}(B) = \mathrm{rk}(AB) = \mathrm{rk}(C) = \mathrm{rk}(CA) = \mathrm{rk}(A) = 2 < \mathrm{rk}(Q) = 3.$$

Consequently, the system of matrix equations $QAB = B, CAQ = C$ *holds, but*

$$QAQ = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\theta} & 0 & \frac{1}{\theta} \\ -\frac{1}{\theta^2} & \frac{1}{\theta} & -\frac{1}{\theta^2} \end{bmatrix} \neq Q.$$

The important requirement in Corollary 11 is $\mathrm{rk}(B) = \mathrm{rk}(C) = \mathrm{rk}(A) = \mathrm{rk}(X)$. *To reduce* $\mathrm{rk}(Q)$ *to* $\mathrm{rk}(A)$ *we use the matrix X obtained by the replacements* $q_{2,3} \to 1/\theta$ *in Q, which gives*

$$X = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\theta} & 0 & \frac{1}{\theta} \\ -\frac{1}{\theta^2} & \frac{1}{\theta} & -\frac{1}{\theta^2} \end{bmatrix}.$$

All requirements in Corollary 11 are satisfied and all the matrix equations $XAX = X$, $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$ *are fulfilled. Furthermore, the matrix equation* $AXA = A$ *is satisfied, which means* $X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

It is important to mention that $B(CAB)^{\dagger}C$ *coincides with X, which is in accordance with the Urquhart representation [36] and its generalizations from [16].*

6. Conclusions

The aim of this paper is to investigate solvability of systems of constrained matrix equations. The main novelty of this paper is the establishment of correlations between solutions of certain constrained matrix equations with corresponding minimization problems. Some well-known results and several new results for the weak Drazin inverse are obtained in particular cases. certain characterizations for the Drazin inverse, group inverse and Moore–Penrose inverse are obtained as corollaries.

Implementation of the stated research highlights can be summarized as follows.

- Conditions (i)–(vi) in Theorem 1 are solutions to (5), while (6) is solved in (12) and (13).
- Conditions (i)–(vi) in Theorem 5 are solutions to (7), while (8) is solved in (17) and (18).
- The unique solution to (9) is $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ and conditions (i)–(vi) in Corollary 11 are conditions for solvability of (9).

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Appendix A

$$
\begin{bmatrix}
(2\epsilon+1)u_{1,1}+\epsilon(u_{2,1}+u_{3,1}) & (2\epsilon+1)u_{1,2}+\epsilon(u_{2,2}+u_{3,2}) & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{2,2}\epsilon+3u_{3,2}\epsilon+\left(-12\epsilon^3+9\epsilon+3\right)u_{1,2}+2\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{\left(24\epsilon^5+28\epsilon^4-2\epsilon^3-17\epsilon^2-8\epsilon-1\right)u_{1,2}+\epsilon\left(-2\epsilon\left(2\epsilon^2+\epsilon+1\right)+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{3,2}\right)}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{(2\epsilon+1)\left(3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon u_{1,1}+(2\epsilon-1)u_{2,1}+\epsilon u_{3,1} & \epsilon u_{1,2}+(2\epsilon-1)u_{2,2}+\epsilon u_{3,2} & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2-6\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}\epsilon+3u_{3,2}\epsilon-3\left(4\epsilon^3-4\epsilon^2-\epsilon+1\right)u_{2,2}+2\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{12u_{3,2}\epsilon^5+8u_{3,2}\epsilon^4+26\epsilon^4-5u_{3,2}\epsilon^3+15\epsilon^3-6u_{3,2}\epsilon^2-9\epsilon^2+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}\epsilon-u_{3,2}\epsilon-7\epsilon+\left(24\epsilon^5+4\epsilon^4-18\epsilon^3-7\epsilon^2+4\epsilon+1\right)u_{2,2}-1}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1\right)+(2\epsilon-1)\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon u_{1,1}+\epsilon u_{2,1}+(2\epsilon+1)u_{3,1} & \epsilon u_{1,2}+\epsilon u_{2,2}+(2\epsilon+1)u_{3,2} & -\frac{(\epsilon-1)\left(12u_{3,2}\epsilon^3+3(2\epsilon+1)u_{1,2}\epsilon^2+3(2\epsilon+1)u_{2,2}\epsilon^2+12u_{3,2}\epsilon^2-6\epsilon^2+3u_{3,2}\epsilon-6\epsilon-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{24u_{3,2}\epsilon^5+28u_{3,2}\epsilon^4-14\epsilon^4-2u_{3,2}\epsilon^3-7\epsilon^3-17u_{3,2}\epsilon^2+4\epsilon^2+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}\epsilon+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}\epsilon-8u_{3,2}\epsilon+\epsilon-u_{3,2}}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)\right)+(2\epsilon+1)\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon(u_{1,1}+u_{2,1}+u_{3,1}) & \epsilon(u_{1,2}+u_{2,2}+u_{3,2}) & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}\epsilon+3\left(-2\epsilon^2+\epsilon+1\right)u_{2,2}\epsilon+3u_{3,2}\epsilon-3\epsilon-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{12u_{3,2}\epsilon^4+8u_{3,2}\epsilon^3+2\epsilon^3-5u_{3,2}\epsilon^2+13\epsilon^2-6u_{3,2}\epsilon+8\epsilon+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}-u_{3,2}+1}{2\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(-6u_{2,1}\epsilon^3-6u_{2,2}\epsilon^3-6u_{3,1}\epsilon^3-6u_{3,2}\epsilon^3+3u_{2,1}\epsilon^2+2u_{2,2}\epsilon^2+3u_{3,1}\epsilon^2+2u_{3,2}\epsilon^2-\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+6u_{2,1}\epsilon+4u_{2,2}\epsilon+6u_{3,1}\epsilon+4u_{3,2}\epsilon-2\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}+u_{2,1}+u_{3,1}-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon(3u_{1,1}+u_{2,1}+u_{3,1}) & \epsilon(3u_{1,2}+u_{2,2}+u_{3,2}) & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2+9\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}\epsilon+3\left(-2\epsilon^2+\epsilon+1\right)u_{2,2}\epsilon+3u_{3,2}\epsilon+3\epsilon-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{12u_{3,2}\epsilon^4+8u_{3,2}\epsilon^3-10\epsilon^3-5u_{3,2}\epsilon^2-5\epsilon^2-6u_{3,2}\epsilon+6\epsilon+3\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}-u_{3,2}+1}{2\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(-6u_{2,1}\epsilon^3-6u_{2,2}\epsilon^3-6u_{3,1}\epsilon^3-6u_{3,2}\epsilon^3+3u_{2,1}\epsilon^2+2u_{2,2}\epsilon^2+3u_{3,1}\epsilon^2+2u_{3,2}\epsilon^2+5\epsilon^2+6\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+6u_{2,1}\epsilon+4u_{2,2}\epsilon+6u_{3,1}\epsilon+4u_{3,2}\epsilon-2\epsilon+\left(-18\epsilon^3+9\epsilon^2+18\epsilon+3\right)u_{1,1}+u_{2,1}+u_{3,1}-3\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1}
\end{bmatrix}.
$$

Appendix B

$$
\begin{bmatrix}
2u_{1,1}+u_{1,2}+u_{1,3} & u_{1,1}+u_{1,3} & u_{1,1}+u_{1,2}+2u_{1,3} & u_{1,1}+u_{1,2}+u_{1,3} & 2u_{1,1}+u_{1,2}+u_{1,3} \\
2u_{2,1}+u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 & u_{2,1}+u_{2,3} & u_{2,1}+2u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 & u_{2,1}+u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 & 2u_{2,1}+u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 \\
\frac{9u_{3,2}\epsilon-11\epsilon+(4-18\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2u_{3,2}+2}{9\epsilon+2} & \frac{-5\epsilon+(2-9\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2}{9\epsilon+2} & \frac{9u_{3,2}\epsilon-4\epsilon+(2-9\epsilon)u_{2,1}+(4-18\epsilon)u_{2,3}+2u_{3,2}+4}{9\epsilon+2} & \frac{9u_{3,2}\epsilon-5\epsilon+(2-9\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2u_{3,2}+2}{9\epsilon+2} & \frac{9u_{3,2}\epsilon-11\epsilon+(4-18\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2u_{3,2}+2}{9\epsilon+2} \\
\frac{2\left(6\epsilon+(9\epsilon+4)u_{2,1}+2\right)}{9\epsilon+2}+\frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{11\epsilon+(9\epsilon+4)u_{2,1}+(9\epsilon+4)u_{2,3}+4}{9\epsilon+2} & \frac{6\epsilon+(9\epsilon+4)u_{2,1}+2}{9\epsilon+2}+\frac{2\left(5\epsilon+(9\epsilon+4)u_{2,3}+2\right)}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{6\epsilon+(9\epsilon+4)u_{2,1}+2}{9\epsilon+2}+\frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{2\left(6\epsilon+(9\epsilon+4)u_{2,1}+2\right)}{9\epsilon+2}+\frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 \\
-u_{1,2}+\frac{2\left(-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1\right)}{9\epsilon+2}+\frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2} & \frac{-9u_{2,1}\epsilon-9u_{2,3}\epsilon-6\epsilon-(9\epsilon+2)u_{1,1}-(9\epsilon+2)u_{1,3}+u_{2,1}+u_{2,3}+1}{9\epsilon+2} & -u_{1,2}+\frac{-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1}{9\epsilon+2}-\frac{2\left(6\epsilon+(9\epsilon+2)u_{1,3}+(9\epsilon-1)u_{2,3}\right)}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2} & -u_{1,2}+\frac{-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1}{9\epsilon+2}+\frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2} & -u_{1,2}+\frac{2\left(-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1\right)}{9\epsilon+2}+\frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2}
\end{bmatrix}.
$$

References

1. Penrose, R. A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.* **1955**, *51*, 406–413.
2. Getson, A.J.; Hsuan, F.C. {2}-*Inverses and Their Statistical Applications*; Lecture Notes in Statistics 47; Springer: Berlin, Germany, 1988.
3. Rao, C.R. A note on a generalized inverse of a matrix with applications to problems in mathematical statistics. *J. R. Soc. Ser. B* **1962**, *24*, 152–158.
4. Ben-Israel, A.; Greville, T.N.E. *Generalized Inverses: Theory and Applications*, 2nd ed.; Springer: New York, NY, USA, 2003.
5. Nashed, M.Z. *Generalized Inverse and Applications*; Academic Press: New York, NY, USA, 1976.
6. Wei, Y. A characterization and representation of the generalized inverse $A^{(2)}_{T,S}$ and its applications. *Linear Algebra Appl.* **1998**, *280*, 87–96.
7. Wei,Y.; Wu, H. The representation and approximation for the generalized inverse $A^{(2)}_{T,S}$. *Appl. Math. Comput.* **2003**, *135*, 263–276.
8. Yang, H.; Liu, D. The representation of generalized inverse $A^{(2)}_{T,S}$ and its applications. *J. Comput. Appl. Math.* **2009**, *224*, 204–209.
9. Zheng, B.; Wang, G. Representation and approximation for generalized inverse $A^{(2)}_{T,S}$: Revisited. *Appl. Math. Comput.* **2006**, *22*, 225–240.
10. Cao, C.G.; Zhang, X. The generalized inverse $A^{(2)}_{T,*}$ and its applications. *J. Appl. Math. Comput.* **2003**, *11*, 155–164.
11. Wang, G.R.; Wei, Y.; Qiao, S. *Generalized Inverses: Theory and Computations*; Science Press: Beijing, China; Springer: Berlin/Heidelberg, Germany , 2018.
12. Wei, Y.; Stanimirović, P.S.; Petković, M. *Numerical and Symbolic Computations of Generalized Inverses*; World Scientific: Singapore, 2018.
13. Sheng, X.; Chen, G. Full-rank representation of generalized inverse $A^{(2)}_{T,S}$ and its applications. *Comput. Math. Appl.* **2007**, *54*, 1422–1430.
14. Sheng, X.; Chen, G.L.; Gong, Y. The representation and computation of generalized inverse $A^{(2)}_{T,S}$. *J. Comput. Appl. Math.* **2008**, *213*, 248–257.
15. Stanimirović, P.S.; Ćirić, M.; Stojanović, I.; Gerontitis, D. Conditions for existence, representations and computation of matrix generalized inverses. *Complexity* **2017**, *2017*, 6429725.
16. Stanimirović, P.S.; Ćirić, M.; Lastra, A.; Sendra, J.R.; Sendra, J. Representations and symbolic computation of generalized inverses over fields. *Appl. Math. Comput.* **2021**, *406*, 126287.
17. Stanimirović, P.S.; Ćirić, M.; Lastra, A.; Sendra, J.R.; Sendra, J. Representations and geometrical properties of generalized inverses over fields. *Linear Multilinear Algebra*. https://doi.org/10.1080/03081087.2021.1985420.
18. Stanimirović, P.S.; Soleymani, F.; Haghani, F.K. Computing outer inverses by scaled matrix iterations. *J. Comput. Appl. Math.* **2016**, *296*, 89–101.
19. Ma, X.; Nashine, H.K.;Shi, S.; Soleymani, F. Exploiting higher computational efficiency index for computing outer generalized inverses. *Appl. Numer. Math.* **2022**, *175*, 18–28.
20. Kansal, M.; Kumar, S.; Kaur, M. An efficient matrix iteration family for finding the generalized outer inverse. *Appl. Math. Comput.* **2022**, *430*, 127292.
21. Petković, M.; Krstić, M.A.; Rajković, K.P. Rapid generalized Schultz iterative methods for the computation of outer inverses. *J. Comput. Appl. Math.* **2018**, *344*, 572–584.
22. Cordero, A.; Soto-Quiros, P.; Torregrosa, J.R. A general class of arbitrary order iterative methods for computing generalized inverses. *Appl. Math. Comput.* **2021**, *409*, 126381.

23. Dehghan, M.; Shirilord, A. A fast computational algorithm for computing outer pseudo-inverses with numerical experiments. *J. Comput. Appl. Math.* **2022**, *408*, 114128.
24. Campbell, S.L.; Meyer, C.D., Jr. *Generalized Inverses of Linear Transformations*; Dover Publications, Inc.: New York, NY, USA, 1991; Corrected Reprint of the 1979 Original; SIAM: Philadelphia, PA, USA, 2008.
25. Levine, J.; Hartwig, R.E. Applications of Drazin inverse to the Hill cryptographic systems. *Cryptologia* **1980**, *1558–1586*, 71–85.
26. Prasad, K.M.; Mohana, K.S. Core-EP inverse. *Linear Multilinear Algebra* **2014**, *62*, 792–802.
27. Baksalary, O.M.; Trenkler, G. Core inverse of matrices. *Linear Multilinear Algebra* **2010**, *58*, 681–697.
28. Malik, S.B.; Thome, N. On a new generalized inverse for matrices of an arbitrary index. *Appl. Math. Comput.* **2014**, *226*, 575–580.
29. Zhou, Y.; Chen, J.; Zhou, M. m-weak group inverses in a ring with involution. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 2* **2021**, *115* 2, 1–13.
30. Wang, H.; Chen, J. Weak group inverse. *Open Math.* **2018**, *16*, 1218–1232.
31. Ferreyra, D.E.; Malik, S.B. A generalization of the group inverse. *Quaest. Math* **2023**. https://doi.org/10.2989/16073606.2022.2144533.
32. Campbell, S.L.; Meyer, C.D. Weak Drazin inverses. *Linear Algebra Appl.* **1978**, *20*, 167–178.
33. Wu, C.; Chen, J. Minimal rank weak Drazin inverses: a class of outer inverses with prescribed range. *Electron. Linear Algebra* **2023**, *39*, 1–16.
34. Mosić, D.; Stanimirović, P.S. Existence and Representation of Solutions to Some Constrained Systems of Matrix Equations. In *Matrix and Operator Equations and Applications*; Book Series: Mathematics Online First Collections; Moslehian, M.S., Ed.; Springer: Cham, Switzerland, 2023. Available online: https://link.springer.com/book/9783031253850 (accessed on 1 January 2023.).
35. Deng, C. On the solutions of operator equation $CAX = C = XAC$. *J. Math. Anal. Appl.* **2013**, *398*, 664–670.
36. Urquhart, N.S. Computation of generalized inverse matrtices which satisfy specified conditions. *SIAM Rev.* **1968**, *10*, 216–218.