Generalized core-EP inverse for square matrices

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Abstract

We introduce, investigate and apply two types of generalized inverses for square complex matrices using an arbitrary inner inverse of the input matrix and its core-EP inverse. Introduced combinations of inner inverses and the core-EP inverse are termed as inner core-EP (ICEP) and core-EP inner (CEPI) inverse. Additionally, we extend the notion of the P-core inverse for square matrices with arbitrary index. A few equivalent characterizations and effective representations of the introduced generalized inverses are suggested. In addition, the representations of these inverses are established via core-EP and HS decompositions. Induced binary relations for these inverses are introduced and their properties are considered. An application of these inverses in solving linear systems also discussed.

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1. Introduction

The matrix generalized inverses and their applications were systematically studied in monographs [5, 37, 41]. Generalized inverses are very useful tools in graph theory, special matrices, difference equations, and singular differential equations (for example; see [6],[8] and [44]). The notion of core inverse was introduced originally by Baksalary and Trenkler in [1, 2] and then extended by the core-EP inverse to square matrices of arbitrary index [25]. Further, it has extended to rectangular matrices [3, 11, 35], and a few representations & characterizations have been discussed in [13, 21, 14, 46, 45]. Prasad and Raj [26] proposed the bordering method [23] for calculating the core-EP inverse. An iterative method for approximating the core-EP inverse was derived by Prasad *et al.* in [24]. The extension of the core-EP inverse to bounded operators in a Hilbert space was studied in [32, 28]. Mosić in [30], introduced the core-EP inverse of elements in a Banach algebra and then extended it to the elements in C^*−algebras [31], tensors [38], the elements in a ring [12, 29], and matrices over the quaternion skew field [20]. Generalized inverses have found applications in solving various optimization optimization and approximation problems. The Moore-Penrose generalized inverse $A^\dagger$ of A generates best approximate solution $A^\dagger b$ of the system of linear equations $Ax = b$, such that $\|Mx - v\| \geq \|MM^\dagger v - v\|$ for arbitrary matrix $M \in \mathbb{C}^{m\times n}$ and arbitrary vector $v \in \mathbb{C}^m$ [36]. Moreover, for the matrix equation $MX = V$it is known that [36] $\|MM^\dagger V - V\| \leq \|MX - V\|$, $\forall X$ and
$\|MM^\dagger V - V\| = \|MX - V\| \Longleftrightarrow X = M^\dagger V + (I - M^\dagger M)Q$, such that Q is arbitrary.
Moreover, $\|M^\dagger V\| \leq \|M^\dagger V + (I - M^\dagger M)Q\|$. These extremely useful results about least squares and best approximate solutions have initiated a large number of researches on the topic of generalized inverses.

A combination of the Moore–Penrose inverse with the core-EP inverse for bounded linear operators in Hilbert space, known as MPCEP, and its dual CEPMP inverses, was introduced in [7]. The authors o that paper have addressed the equivalent relationships among Moore-Penrose, generalized Drazin (gD), gDMP, MPgD, and core-EP inverses using appropriate idempotents. Behera *et al.* [4] presented W-weighted MP core-EP inverse and its dual for rectangular matrices. The notion of the DMP inverse and its properties for square matrices was introduced by Malik it et al. in [22] by combining the Drazin and MP inverse. On the other hand, the MPD inverse for square matrices was studied in [20]. In [27], Mehdipour and Salemi introduced CMP inverse by combining the core part of a square matrix and its MP inverse. The expressions of MP1 and MP2 inverse for rectangular matrices, characterizations, and applications, as well as for their duals, were given by Hernández *et al.* in [17, 18] respectively. The OMP, MPO, and MPOMP inverses for rectangular matrices were discussed in [33]. Using GD and MP inverse Hernández et al. [16] introduced the properties and characterizations of GDMP-inverse and its dual for square matrices. The additive property, forward, and reverse order laws for GDMP, and W-weighted GDMP inverses are recently investigated by Amit *et al.* [19].

Motivated by the work of [16, 17, 18], in in current research we establish and examine main propertied and representations of inner core-EP (ICEP), its dual termed as core-EP inner (CEPI) inverses, and P-core-EP inverse. The main contributions of the manuscript are as follows.

- We introduce two classes of generalized inverses: ICEP inverse (inner and core-EP inverse), its dual called CEPI inverse, and P-core-EP inverse.
- A few representations and characterizations of these inverses are investigated.
- Representations of theses inverses based on core-nilpotent decomposition and the Hartwig and Spindelböck decomposition are established.
- A binary relation based on introduced inverses is derived and its properties are considered.

The overall organization of sections is as follows. Introduction to basic results and motivation for the research are presented in Section 1. Some preliminaries are presented in Section 2. The class of ICEP Inverses is introduced and investigated in Section 3. A relation defined on the class of matrices using ICEP inverse is introduces in the same section. The notion of the CEPI inverse is introduced in Section 4. Section 5 introduces and investigates an extension of the P-core inverse for matrices of arbitrary index in the form of a new generalized inverse which is called the P-core-EP inverse. Section 6 considers the solution of linear certain equations in terms of ICEP inverses, CEPI inverses and the P-core-EP inverse. Some concluding remarks and vision of further research are given in the last section.

2. Preliminaries

First, we will discuss a few useful notations and definitions of some generalized inverses. The set of all complex $m \times n$ matrices will be marked with $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, the conjugate transpose, range space, and null space of A, respectively denoted as A^*, $\mathcal{R}(A)$, and $\mathcal{N}(A)$. We denote an $n \times n$ identity matrix by I_n and an orthogonal projection on $\mathcal{R}(A)$ by P_A. The index of a matrix A means the smallest non-negative integer $k = \text{ind}(A)$ satisfying $\text{rank}(A^k) = \text{rank}(A^{k+1})$. Next, we will define the generalized inverse using the matrix equations as given in Table 1. Let Λ be a nonempty subset of elements from $\{1, 2, 3, 4, 5, 1^k, 6, \ldots\}$.

Table 1: Matrix equations for defining generalized inverses

Label	(1)	(2)	(3)	(4)
Equation	$AZA = A$	$ZAZ = A$	$(AZ)^* = AZ$	$(ZA)^* = ZA$
Label	(5)	(1^k)	(6)	
Equation	$AZ = ZA$	$ZA^{k+1} = A^k$	$AZ^2 = Z$	

If a matrix $Z \in \mathbb{C}^{n\times m}$ satisfies equation (i) for each $i \in \Lambda$ then Z is called a $\{\Lambda\}$-inverse of A. We denote such an inverse by $A^{(\lambda)}$ and set of all $\{\Lambda\}$-inverse of A by $A\{\Lambda\}$. Using these representations, we now restate definitions of the following generalized inverses.

Definition 2.1. [5] *Let $A \in \mathbb{C}^{m\times n}$. A matrix Z is called:*

(a) *An inner inverse of A if it fulfills equation* (1) *and denoted by A^{-}.*
(b) *An outer inverse of A if $Z \in A\{2\}$ and denoted as $A^{=}$. Further, if $\mathcal{R}(Z) = L$ and $\mathcal{N}(Z) = M$ then we denote it by $A^{(2)}_{L,M}$.*
(c) *The Moore-Penrose inverse $A^{\dagger}$ of A if $Z \in A\{1, 2, 3, 4\}$* [36].

Definition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. A matrix Z is called:*

(a) *the Drazin inverse of A if $Z \in A\{2, 5, 1^k\}$ and denoted by A^{D}* [10]*;*
(b) *the core-EP inverse of A if $Z \in A\{2, 1^k, 6\}$ and denoted by $A^{\textcircled{\dagger}}$* [25].

Lemma 2.1. ([25]) *Let $A \in \mathbb{C}^{n\times n}$. Then the Core-EP inverse satisfies the following :*

$$A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} \text{ and } \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}((A^{\textcircled{\dagger}})^{*}) = \mathcal{R}(A^{k}).$$

Lemma 2.2. ([12]) *Let $A \in \mathbb{C}^{n\times n}$. For $l \geqslant k$, the core-EP inverse of A is given by*

$$A^{\textcircled{\dagger}} = A^{\mathrm{D}}A^{l}(A^{l})^{\dagger} = A^{l}\left(A^{l+1}\right)^{\dagger}.$$

A few composite inverses have been studied in recent years, as restated below in definitions 2.3 and 2.4.

Definition 2.3. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. Then*

(a) *the* DMP *inverse of A is denoted by $A^{\mathrm{D},\dagger}$ and defined as $A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^{\dagger}$* [22].
(b) *the* MPD *inverse of A is defined as $A^{\dagger,\mathrm{D}} = A^{\dagger}AA^{\mathrm{D}}$* [22].
(c) *the* CMP *inverse of A is denoted as $A^{c,\dagger}$ and defined by $A^{\dagger}AA^{\mathrm{D}}AA^{\dagger}$.*
(d) *the* MPCEP *inverse of A is defined as $A^{\dagger,\textcircled{\dagger}} = A^{\dagger}AA^{\textcircled{\dagger}}$* [7, 34].
(e) *the $*$CEPMP is defined as $A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}}AA^{\dagger}$,* [7].

Definition 2.4. [33] *Let $A \in \mathbb{C}^{m\times n}$ and suppose the existence of $A^{(2)}_{T,S}$ exists.*

(a) *The* OMP *inverse of A is defined as $A^{(2),\dagger}_{T,S} = A^{(2)}_{T,S}AA^{\dagger}$.*
(b) *The* MPO *inverse of A is defined as $A^{\dagger,(2)}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}$.*
(c) *The* MPOMP *inverse of A is defined as $A^{\dagger,(2),\dagger}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}AA^{\dagger}$.*

Based on the presented representations of composite generalized inverses, we note that combinations of outer inverses were a popular topic of scientific research. The subject of our research in this paper is combinations of inner generalized inverses with outer inverses. Particularly, we investigate combinations of inner inverses with the core-EP inverse.

3. ICEP Inverses

In this section, we define ICEP inverses for square matrices and discuss a few representations and characterizations of these inverses. To simplify presentation, $A \in \mathbb{C}^{n\times n}$ will be supposed and an arbitrary $G \in A\{1\}$ is considered. Such environment situation will be denoted by $A \triangleright_n G$. For the remaining of this paper, unless otherwise specified, it is understood that all input matrices A will be square and of index k.

Theorem 3.1. *For $A \triangleright_n G$, the matrix $Z = GAA^{\textcircled{\dagger}}$ is the unique solution to the matrix equations*

$$ZAZ = Z, \quad ZA = GAA^{\textcircled{\dagger}}A, \quad \text{and} \quad AZ = AA^{\textcircled{\dagger}}. \tag{3.1}$$

Proof. Let $Z = GAA^{\textcircled{\dagger}}$. Then $AZ = AGAA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}$, $ZA = GAA^{\textcircled{\dagger}}A$, and

$$ZAZ = GAA^{\textcircled{\dagger}}AGAA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = Z.$$

Thus Z satisfies (3.1). Suppose Z_1 and Z_2 are two solutions of the system (3.1). From $AZ_1 = AA^{\textcircled{\dagger}} = AZ_2$ and $Z_1A = GAA^{\textcircled{\dagger}}A = Z_2A$ we get

$$Z_1 = Z_1(AZ_1) = Z_1(AZ_2) = (Z_1A)Z_2 = (GAA^{\textcircled{\dagger}}A)Z_2 = Z_2AZ_2 = Z_2.$$

Hence Z is the unique solution to (3.1). $\square$

In the light of Theorem 3.1, we define the ICEP inverse of a square matrix as given below.

Definition 3.1. *For $A \rhd_n G$, the* ICEP *inverse of A is denoted by $A_G^{-,\textcircled{\dagger}}$ and defined as $A_G^{-,\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}$.*

Remark 3.1. *Notice that every fixed inner inverse $G \in A\{1\}$ of A may give rise to a different* ICEP *inverse of A. Henceforth, if we mention the* ICEP *inverse of A, then it includes previously fixed inner inverse. The set of all* ICEP *inverses of A is defined by $A\{-,\textcircled{\dagger}\} = \{GAA^{\textcircled{\dagger}} \mid G \in A\{1\}\} = A\{1\}AA^{\textcircled{\dagger}}$.*

Example 3.1 verifies that the ICEP inverse of a matrix $A \in \mathbb{C}^{n\times n}$ differs from composite generalized inverses defined so far.

Example 3.1. Let $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Clearly, $\mathrm{ind}(A) = 1$, which leads to the calculation

$$A^\dagger = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}, \quad A^{\mathrm{D}} = A\left(A^3\right)^\dagger A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\textcircled{\dagger}} = A\left(A^2\right)^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Further evaluation gives

$$A^{c,\dagger} = A^\dagger AA^{\mathrm{D}}AA^\dagger = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}, \quad A^{\dagger,\mathrm{D}} = A^\dagger AA^{\mathrm{D}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^\dagger A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{\dagger,\textcircled{\dagger}} = A^\dagger AA^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}, \quad A^{\textcircled{\dagger},\dagger} = A^{\textcircled{\dagger}}AA^\dagger = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For a fixed inner inverse $G = \begin{bmatrix} 1 & 1 & 5 & 2 \\ 1 & 6 & 0 & 9 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 4 & -2 \end{bmatrix} \in A\{1\}$, further calculation gives

$$A^{\mathrm{D},-} = A^{\mathrm{D}}AG = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{-,\mathrm{D}} = GAA^{\mathrm{D}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_G^{-,\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, it is easy to verify $A_G^{-,\textcircled{\dagger}} \notin \{A^{-,\mathrm{D}}, A^{\dagger,\textcircled{\dagger}}, A^{\dagger,\mathrm{D}}, A^{\mathrm{D},\dagger}, A^{c,\dagger}, A^{\textcircled{\dagger},\dagger}\}$.

Example 3.2. For the matrix A from Example 3.1, the general solution to the matrix equation $AXA = A$ gives the set of inner inverses as

$$A\{1\} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ 0 & 1 & x_{3,3} & x_{3,4} \\ 1 - x_{1,1} & -x_{1,2} & x_{4,3} & x_{4,4} \end{bmatrix} \mid x_{ij} \in \mathbb{C} \right\}.$$

Then

$$A\{-, \textcircled{\dagger}\} = A\{1\}AA^{\textcircled{\dagger}} = \left\{ \begin{bmatrix} x_{1,1} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 - x_{1,1} & 0 & 0 & 0 \end{bmatrix} \mid x_{ij} \in \mathbb{C} \right\}.$$

Clearly, $A_G^{-,\textcircled{\dagger}}$ from Example 3.1 can be obtained in the particular case $x_{11} = x_{21} = 1$ in $A\{-, \textcircled{\dagger}\}$.

Now, we derive an expression for the ICEP inverse with the fixed inner inverse G via the core-EP decomposition. The core-EP decomposition [42] of $A \in \mathbb{C}^{n\times n}$ is equal to

$$A = U \begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^*, \tag{3.2}$$

where U is a unitary matrix, $T_1 \in \mathbb{C}^{r\times r}$ is a non-singular matrix, and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent. Now, we express an inner inverse of $A \in \mathbb{C}^{n\times n}$ as

$$G = U \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} U^*, \tag{3.3}$$

where $T_1Z_1 + T_2Z_3 = I_r$, $(T_1Z_2 + T_2Z_4)N = 0$, $NZ_3 = 0$, and $Z_4 \in N\{1\}$. In [42], the core-EP inverse of A is expressed as

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{3.4}$$

Theorem 3.2 is stated applying decompositions of A, G and $A^{\textcircled{\dagger}}$.

Theorem 3.2. $A \rhd_n G$. *Consider the decompositions of* A, G, *and* $A^{\textcircled{\dagger}}$ *respectively as given in* (3.2), (3.3), *and* (3.4), *respectively. Then ICEP inverse of* A *is represented as*

$$A_G^{-,\textcircled{\dagger}} = U \begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix} U^*,$$

where $T_1Z_1 + T_2Z_3 = I_r$ *and* $NZ_3 = 0$. *Consequently,* $A_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}}$ *if and only if* $Z_3 = 0$.

3.1. Characterization of ICEP inverses

We will discuss a few characterizations of ICEP inverses and their relationship with other classes of outer inverses.

Corollary 3.1. *Let* $A \rhd_n G$ *and* $l \geqslant k = \mathrm{ind}(A)$ *be an arbitrary integer. The following statements are valid in such circumstances:*

(i) $A_G^{-,\textcircled{\dagger}} = GA^l(A^l)^\dagger = GP_{\mathcal{R}(A^l)}$;

(ii) $A_G^{-,\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(GA^l),\ \mathcal{N}((A^l)^*)} = A^{(2)}_{\mathcal{R}(GA^l(A^l)^*),\ \mathcal{N}(GA^l(A^l)^*)}$;

(iii) $A^{l+1}A_G^{-,\textcircled{\dagger}} = A^{2l}(A^l)^\dagger$;

(iv) $A_G^{-,\textcircled{\dagger}} A^l = GA^l$;

(v) $A_G^{-,\textcircled{\dagger}} = GA^l\left((A^l)^*AGA^l\right)^\dagger (A^l)^* = GA^l\left((A^l)^*A^{l-1}\right)^\dagger (A^l)^* \iff \mathrm{rank}((A^l)^*A^{l-1}) = \mathrm{rank}((A^l)^*) = \mathrm{rank}(GA^l)$.

Proof. (i) Choose $l \geq k$. Then by Lemma 2.2, we have $A^{\textcircled{\dagger}} = A^{\mathrm{D}}A^l(A^l)^\dagger$. Now

$$A_G^{-,\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = GAA^{\mathrm{D}}A^l(A^l)^\dagger = GA^l(A^l)^\dagger.$$

Using $P_{\mathcal{R}(A^l)} = A^l(A^l)^\dagger$, we conclude $GA^l(A^l)^\dagger = GP_{\mathcal{R}(A^l)}$.
(ii) From part (i), we get $A_G^{-,\textcircled{\dagger}} = GA^l(A^l)^\dagger$. Now

$$\mathcal{R}(GA^l) = \mathcal{R}\left(GA^l(A^l)^*\right) = \mathcal{R}\left(GA^l(A^l)^\dagger\right) = \mathcal{R}(A_G^{-,\textcircled{\dagger}}).$$

Further it follows

$$(A^l)^* = (A^l(A^l)^\dagger A^l)^* = (A^l)^*A^l(A^l)^\dagger = (A^l)^*AGA^l(A^l)^\dagger = (A^l)^*AA_G^{-,\textcircled{\dagger}}$$

and

$$A_G^{-,\textcircled{\dagger}} = GA^l(A^l)^\dagger = G(A^l(A^l)^\dagger)^* = G[(A^l)^\dagger]^*(A^l)^*$$

Thus we get $\mathcal{N}\left((A^l)^*\right) = \mathcal{N}(A_G^{-,\textcircled{\dagger}})$.
(iii) $A^{l+1}A_G^{-,\textcircled{\dagger}} = (A^{l+1}GA)A^{\textcircled{\dagger}} = A^{l+1}A^{\mathrm{D}}A^l(A^l)^\dagger = A^{2l}(A^l)^\dagger$.
(iv) Using $A^{\textcircled{\dagger}} = A^{\mathrm{D}}A^l(A^l)^\dagger$, we calculate

$$A_G^{-,\textcircled{\dagger}}A^l = GAA^{\textcircled{\dagger}}A^l = GAA^{\mathrm{D}}A^l(A^l)^\dagger A^l = GAA^{\mathrm{D}}A^l = GA^l.$$

(v) The representations in this statement are derived from the representation of outer inverses

$$B(CAB)^\dagger C = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} \iff \mathrm{rank}(CAB) = \mathrm{rank}(B) = \mathrm{rank}(C),$$

given by the Urquhart representation [40] and its extensions given in [39]. □

Example 3.3. Consider the input matrix of index $\mathrm{ind}(A) = 3$:

$$A = \begin{bmatrix} a & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

such that a, b are two left unassigned symbols with values from the set of complex numbers $\mathbb{C}$. The set of inner inverses is equal to

$$A\{1\} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & \frac{1}{a} & 0 & -\frac{1}{b} & 0 & x_{2,6} \\ 0 & 0 & \frac{1}{a} & 0 & -\frac{1}{b} & x_{3,6} \\ \frac{1}{a} - x_{1,1} & -x_{1,2} & -x_{1,3} & -x_{1,4} & -x_{1,5} & x_{4,6} \\ 0 & 0 & 0 & \frac{1}{b} & 0 & x_{5,6} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & x_{6,6} \end{bmatrix} \mid x_{ij} \in \mathbb{C} \right\}.$$

Then

$$A^{\textcircled{\dagger}} = A^3\left(A^4\right)^\dagger = \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which initiates

$$A\{-,\textcircled{\dagger}\} = A\{1\}AA^{\textcircled{\dagger}} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a} & 0 & 0 & 0 \\ \frac{1}{a} - x_{1,1} & -x_{1,2} & -x_{1,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid x_{1j} \in \mathbb{C} \right\}.$$

Verification based on the symbolic calculus gives $A\{1\}A^3\left(A^3\right)^{\dagger} = A\{-,\textcircled{\dagger}\}$, which is a confirmation of Corollary 3.1.

Example 3.4. This example is aimed to further verification of Corollary 3.1. The input matrix

$$A = \begin{bmatrix} \frac{\zeta}{2} & 0 & 0 & 0 & \frac{\zeta}{2} & 0 & 0 & 0 \\ 0 & \frac{\zeta}{2} & 0 & 0 & 0 & \frac{\zeta}{2} & 0 & 0 \\ 0 & 0 & \frac{\zeta}{2} & 0 & 0 & 0 & \frac{\zeta}{2} & 0 \\ 0 & 0 & 0 & \frac{\zeta}{2} & 0 & 0 & 0 & \frac{\zeta}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\eta}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\eta}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where ζ, η are two left unassigned symbols with domains in the set of complex numbers $\mathbb{C}$. Inner inverses of A are defined by

$$A\{1\} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} \\ 0 & \frac{2}{\zeta} & 0 & 0 & -\frac{2}{\eta} & 0 & 0 & x_{2,8} \\ 0 & 0 & \frac{2}{\zeta} & 0 & 0 & -\frac{2}{\eta} & 0 & x_{3,8} \\ 0 & 0 & 0 & \frac{2}{\zeta} & 0 & 0 & -\frac{2}{\eta} & x_{4,8} \\ \frac{2}{\zeta} - x_{1,1} & -x_{1,2} & -x_{1,3} & -x_{1,4} & -x_{1,5} & -x_{1,6} & -x_{1,7} & x_{5,8} \\ 0 & 0 & 0 & 0 & \frac{2}{\eta} & 0 & 0 & x_{6,8} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{\eta} & 0 & x_{7,8} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\eta} & x_{8,8} \end{bmatrix} \mid x_{ij} \in \mathbb{C} \right\}.$$

Since $\operatorname{ind}(A) = 4$, symbolic calculation gives

$$A^{\textcircled{\dagger}} = A^4\left(A^5\right)^{\dagger} = \begin{bmatrix} \frac{2}{\zeta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\zeta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\zeta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\zeta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Further calculation generates the set of ICEP inverses:

$$A\{-,\textcircled{\dagger}\} = A\{1\}AA^{\textcircled{\dagger}} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\zeta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\zeta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\zeta} & 0 & 0 & 0 & 0 \\ \frac{2}{\zeta} - x_{1,1} & -x_{1,2} & -x_{1,3} & -x_{1,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid x_{1j} \in \mathbb{C} \right\}.$$

The identity $A\{1\}A^3\left(A^3\right)^{\dagger} = A\{-,\textcircled{\dagger}\}$ gives another confirmation of Corollary 3.1, part (i).

To confirm the statement (iii) of Corollary 3.1 we calculate

$$A^5 \cdot A\{-,\textcircled{\dagger}\} = \left\{ \begin{bmatrix} \frac{\zeta^4}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^4}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^4}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^4}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid x_{1j} \in \mathbb{C} \right\}$$

and verify $A^5\, A\{-,\textcircled{\dagger}\} = A^8\left(A^4\right)^{\dagger}$.

The statement (iv) of Corollary 3.1 follows from

$$A\{-,\textcircled{\dagger}\}\, A^4 = \left[\begin{matrix} \frac{1}{16}\zeta^4 x_{1,1} & \frac{1}{16}\zeta^4 x_{1,2} & \frac{1}{16}\zeta^4 x_{1,3} & \frac{1}{16}\zeta^4 x_{1,4} \\ 0 & \frac{\zeta^3}{8} & 0 & 0 \\ 0 & 0 & \frac{\zeta^3}{8} & 0 \\ 0 & 0 & 0 & \frac{\zeta^3}{8} \\ -\frac{1}{16}\zeta^3\left(\zeta x_{1,1}-2\right) & -\frac{1}{16}\zeta^4 x_{1,2} & -\frac{1}{16}\zeta^4 x_{1,3} & -\frac{1}{16}\zeta^4 x_{1,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right.$$

$$\begin{matrix} \frac{1}{16}\zeta^4 x_{1,1} & \frac{1}{16}\zeta^3\left(\eta x_{1,1}+\zeta x_{1,2}\right) & \frac{1}{16}\zeta^2\left(x_{1,1}\eta^2+\zeta\left(\eta x_{1,2}+\zeta x_{1,3}\right)\right) \\ 0 & \frac{\zeta^3}{8} & \frac{\zeta^2\eta}{8} \\ 0 & 0 & \frac{\zeta^3}{8} \\ 0 & 0 & 0 \\ \frac{1}{16}\zeta^3\left(\zeta x_{1,1}-2\right) & -\frac{1}{16}\zeta^2\left(x_{1,2}\zeta^2+\eta x_{1,1}\zeta-2\eta\right) & -\frac{1}{16}\zeta\left(x_{1,3}\zeta^3+\eta x_{1,2}\zeta^2+\eta^2 x_{1,1}\zeta-2\eta^2\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\left. \begin{matrix} \frac{1}{16}\zeta\left(x_{1,1}\eta^3+\zeta\left(x_{1,2}\eta^2+\zeta\left(\eta x_{1,3}+\zeta x_{1,4}\right)\right)\right) \\ \frac{\zeta\eta^2}{8} \\ \frac{\zeta^2\eta}{8} \\ \frac{\zeta^3}{8} \\ \frac{1}{16}\left(-x_{1,4}\zeta^4-\eta x_{1,3}\zeta^3-\eta^2 x_{1,2}\zeta^2-\eta^3 x_{1,1}\zeta+2\eta^3\right) \\ 0 \\ 0 \\ 0 \end{matrix} \right]$$

$$= A\{1\}A^4.$$

Corollary 3.2. *The subsequent representations are valid under the assumption $A \vartriangleright_n G$:*

(i) $AA_G^{-,\textcircled{\dagger}}$ *is an orthogonal projector onto $\mathcal{R}(A^k)$;*

(ii) $A_G^{-,\textcircled{\dagger}}A$ *is a projector onto $\mathcal{R}(G(A^{\textcircled{\dagger}})^*)$ along $\mathcal{N}(A^{\textcircled{\dagger}}A)$.*

Proof. (i) This statement follows from

$$AA_G^{-,\textcircled{\dagger}} = AGAA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}} \text{ and } \mathcal{R}(A^k) = \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}(AA^{\textcircled{\dagger}}) = \mathcal{R}(AA_G^{-,\textcircled{\dagger}}).$$

(ii) Clearly $A_G^{-,\textcircled{\dagger}} AA_G^{-,\textcircled{\dagger}} A = GAA^{\textcircled{\dagger}}A = A_G^{-,\textcircled{\dagger}} A$. Further,

$$\mathcal{R}(G(A^{\textcircled{\dagger}})^*) \supseteq \mathcal{R}(G(A^{\textcircled{\dagger}})^*A^*A) = \mathcal{R}(GAA^{\textcircled{\dagger}}A) = \mathcal{R}(A_G^{-,\textcircled{\dagger}} A)$$

and

$$\begin{aligned}\mathcal{R}(G(A^{\textcircled{\dagger}})^*) &= \mathcal{R}(G(A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}})^*) = \mathcal{R}(G(AA^{\textcircled{\dagger}})^*(A^{\textcircled{\dagger}})^*) = \mathcal{R}(GAA^{\textcircled{\dagger}}(A^{\textcircled{\dagger}})^*)\\ &\subseteq \mathcal{R}(GAA^{\textcircled{\dagger}}) = \mathcal{R}(GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}) \subseteq \mathcal{R}(A_G^{-,\textcircled{\dagger}} A).\end{aligned}$$

Thus, $\mathcal{R}(A_G^{-,\textcircled{\dagger}} A) = \mathcal{R}(G(A^{\textcircled{\dagger}})^*)$. From

$$\begin{aligned}\mathcal{N}(A_G^{-,\textcircled{\dagger}} A) &\subseteq \mathcal{N}(A^{\textcircled{\dagger}}AGAA^{\textcircled{\dagger}}A) = \mathcal{N}(A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A)\\ &= \mathcal{N}(A^{\textcircled{\dagger}}A) \subseteq \mathcal{N}(GAA^{\textcircled{\dagger}}A)\\ &= \mathcal{N}(A_G^{-,\textcircled{\dagger}} A),\end{aligned}$$

we obtain $\mathcal{N}(A_G^{-,\textcircled{\dagger}} A) = \mathcal{N}(A^{\textcircled{\dagger}}A)$ and hence complete the proof. □

Algebraic characterizations of the ICEP inverses are discussed in Theorem 3.3.

Theorem 3.3. *The following characterizations are mutually equivalent:*

(i) $A_G^{-,\textcircled{\dagger}} = Z$.
(ii) $Z = ZAZ,\ AZ = AA^{\textcircled{\dagger}},\ AZA = AA^{\textcircled{\dagger}}A$ *and* $ZA = GAA^{\textcircled{\dagger}}A$.
(iii) $AZ = AA^{\textcircled{\dagger}}$ *and* $GAA^{\textcircled{\dagger}}AZ = Z$.
(iv) $ZA = GAA^{\textcircled{\dagger}}A$ *and* $ZAA^{\textcircled{\dagger}} = Z$.
(v) $ZAA^{\textcircled{\dagger}}A = GAA^{\textcircled{\dagger}}A,\ AA^{\textcircled{\dagger}}AZ = AA^{\textcircled{\dagger}},\ AA^{\textcircled{\dagger}}AZAA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$ *and* $ZAA^{\textcircled{\dagger}}AZ = Z$.
(vi) $ZAA^{\textcircled{\dagger}}A = GAA^{\textcircled{\dagger}}A,\ ZAA^{\textcircled{\dagger}}AZ = Z$ *and*
$AA^{\textcircled{\dagger}}AZ = AA^{\textcircled{\dagger}}$.

Proof. (i)⇒(ii) The implication follows from $AZA = AGAA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$.
(ii)⇒(iii) From $ZA = GAA^{\textcircled{\dagger}}A$, we obtain $GAA^{\textcircled{\dagger}}AZ = (ZA)Z = Z$.
(iii)⇒(i) It follows from $GAA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}(AZ) = Z$.
(ii)⇒(iv) It is sufficient to show $XAA^{\textcircled{\dagger}} = Z$ and it follows from $AA^{\textcircled{\dagger}} = AZ$ and $ZAZ = Z$.
(iv)⇒(i) Based on $ZA = GAA^{\textcircled{\dagger}}A$, it follows $GAA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = (ZA)A^{\textcircled{\dagger}} = Z$.
(i)⇒(iv) From $GAA^{\textcircled{\dagger}} = Z$, we obtain $AA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}AGAA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}AZ$. Now

$$Z = GAA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = ZAA^{\textcircled{\dagger}}AZ,\ AA^{\textcircled{\dagger}}AZAA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$$

and $ZAA^{\textcircled{\dagger}}A = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = GAA^{\textcircled{\dagger}}A$.
(v)⇒(vi) It is obvious.
(vi)⇒(i) It follows from the transformation

$$GAA^{\textcircled{\dagger}} = (GAA^{\textcircled{\dagger}}A)A^{\textcircled{\dagger}} = ZAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = Z(AA^{\textcircled{\dagger}}) = ZAA^{\textcircled{\dagger}}AZ = Z.$$

The proof is completed. □

Next, we discuss an equivalent definition of ICEP inverses in the result given below.

Theorem 3.4. *For $A \triangleright_n G$, the* ICEP *inverse $Z = A_G^{-,\textcircled{\dagger}}$ is the unique solution to the constrained matrix equation*

$$AZ = P_{\mathcal{R}(AA^{\textcircled{\dagger}})} \text{ and } \mathcal{R}(Z) \subseteq \mathcal{R}(GA). \tag{3.5}$$

Proof. From the Corollary 3.2, it follows $AA_G^{-,\textcircled{\dagger}} = P_{\mathcal{R}(AA^{\textcircled{\dagger}})}$. Now $\mathcal{R}(A_G^{-,\textcircled{\dagger}}) = R(GAA^{\textcircled{\dagger}}) \subseteq R(GA)$. Thus $A_G^{-,\textcircled{\dagger}}$ satisfies equation (3.5). It remains to confirm uniqueness of the solution to the system (3.5). Suppose two solutions exist, Z_1 and Z_2, which satisfy equation (3.5). Then $A(Z_1 - Z_2) = P_{\mathcal{R}(AA^{\textcircled{\dagger}})} - P_{\mathcal{R}(AA^{\textcircled{\dagger}})} = 0$. Consequently $\mathcal{R}(Z_1 - Z_2) \subseteq \mathcal{N}(A) = \mathcal{N}(GA)$. Further, from

$$\mathcal{R}(Z_1) \subseteq \mathcal{R}(GA) \text{ and } \mathcal{R}(Z_2) \subseteq \mathcal{R}(GA),$$

we obtain $\mathcal{R}(Z_1 - Z_2) \subseteq \mathcal{R}(GA) \cap \mathcal{N}(GA) = \{0\}$. Therefore, $Z_1 = Z_2$, which completes the proof. □

Next, we derive the formula for ICEP inverses using the projections $I_n - AA_G^{-,\textcircled{\dagger}}$ and $I_n - A_G^{-,\textcircled{\dagger}} A$.

Theorem 3.5. *Suppose $P' = I_n - A_G^{-,\textcircled{\dagger}} A$ and $P'' = I_n - AA_G^{-,\textcircled{\dagger}}$. Then, for $A + P''$ and $A - P''$ both invertible,*

$$A_G^{-,\textcircled{\dagger}} = (I_n - P')(A \pm P'')^{-1}(I_n - P'').$$

Proof. Suppose $A \in \mathbb{C}^{n\times n}$ as in (3.2). By using Theorem(3.2), we can verify that

$$\begin{aligned} P'' &= I_n - AA_G^{-,\textcircled{\dagger}} \\ &= I_n - U\begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^*U\begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix} U^* \\ &= I_n - U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* = U\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^*. \end{aligned}$$

Now

$$A \pm P'' = U\begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^* \pm U\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^* = U\begin{bmatrix} T_1 & T_2 \\ 0 & N \pm I_{n-r} \end{bmatrix} U^*.$$

Since both T_1 and $N \pm I_{n-r}$ invertible, then

$$(A \pm P'')^{-1} = U\begin{bmatrix} T_1^{-1} & -T_1^{-1}T_2(N \pm I_{n-r})^{-1} \\ 0 & (N \pm I_{n-r})^{-1} \end{bmatrix} U^*.$$

Again from

$$I_n - P' = A_G^{-,\textcircled{\dagger}} A = U\begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix} U^*U\begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^* = U\begin{bmatrix} Z_1T_1 & Z_1T_2 \\ Z_3T_1 & Z_3T_2 \end{bmatrix} U^*,$$

and

$$\begin{aligned} (A \pm P'')^{-1}(I_n - P'') &= U\begin{bmatrix} T_1^{-1} & -T_1^{-1}T_2(N \pm I_{n-r})^{-1} \\ 0 & (N \pm I_{n-r})^{-1} \end{bmatrix} U^*U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \end{aligned}$$

we obtain

$$\begin{aligned} (I_n - P')(A \pm P'')^{-1}(I_n - P'') &= U\begin{bmatrix} Z_1T_1 & Z_1T_2 \\ Z_3T_1 & Z_3T_2 \end{bmatrix} U^*U\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U\begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix} U^* \\ &= A_G^{-,\textcircled{\dagger}}, \end{aligned}$$

which finishes the proof. □

The relationship of ICEP inverses with the inverse of a particular bordered nonsingular matrix is analysed in Theorem 3.6.

Theorem 3.6. *For $A \triangleright_n G$, consider full column rank matrices E and F^* satisfying*

$$\mathcal{R}(GA^k) = \mathcal{N}(F) \text{ and } \mathcal{N}((A^k)^*) = \mathcal{R}(E).$$

Then

$$Z = \begin{bmatrix} A & E^\dagger \\ F & 0 \end{bmatrix}$$

is the nonsingular bordered matrix with

$$Z^{-1} = \begin{bmatrix} A_G^{-,\textcircled{\dagger}} & (I_n - A_G^{-,\textcircled{\dagger}} A)F^\dagger \\ E^\dagger(I_n - AA_G^{-,\textcircled{\dagger}}) & -E^\dagger(A - AA_G^{-,\textcircled{\dagger}} A)F^\dagger \end{bmatrix}. \tag{3.6}$$

Proof. Based on Corollary 3.1, we obtain $\mathcal{R}(A_G^{-,\textcircled{\dagger}}) = \mathcal{R}(GA^k) = \mathcal{N}(F)$ and hence $VA_G^{-,\textcircled{\dagger}} = 0$. From

$$\begin{aligned} \mathcal{R}(I_n - AA_G^{-,\textcircled{\dagger}}) &= \mathcal{N}(AA_G^{-,\textcircled{\dagger}}) = \mathcal{N}(A_G^{-,\textcircled{\dagger}}) = \mathcal{N}((A^k)^*) \\ &= \mathcal{R}(E) = \mathcal{R}(EE^\dagger) \\ &= \mathcal{N}(I_n - EE^\dagger), \end{aligned}$$

we get $(I_n - EE^\dagger)(I_n - AA_G^{-,\textcircled{\dagger}}) = 0$ which implies

$$EE^\dagger(I_n - AA_G^{-,\textcircled{\dagger}}) = I_n - AA_G^{-,\textcircled{\dagger}}.$$

Suppose matrix X is the right side of the expression (3.6). Thus,

$$\begin{aligned} ZX &= \begin{bmatrix} AA_G^{-,\textcircled{\dagger}} + EE^\dagger(I_n - AA_G^{-,\textcircled{\dagger}}) & A(I_n - A_G^{-,\textcircled{\dagger}} A)F^\dagger - EE^\dagger(A - AA_G^{-,\textcircled{\dagger}} A)F^\dagger \\ FA_G^{-,\textcircled{\dagger}} & F(I_n - A_G^{-,\textcircled{\dagger}} A)F^\dagger \end{bmatrix} \\ &= \begin{bmatrix} AA_G^{-,\textcircled{\dagger}} + I_n - AA_G^{-,\textcircled{\dagger}} & (I_n - AA_G^{-,\textcircled{\dagger}})AF^\dagger - (I_n - AA_G^{-,\textcircled{\dagger}})AF^\dagger \\ 0 & FF^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ 0 & FF^\dagger \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \\ &= I_{2n}. \end{aligned}$$

Similarly we can show $XZ = I_{2n}$. Therefore, $X = Z^{-1}$. □

A few equivalent characterizations of ICEP inverses are discussed in the subsequent results.

Theorem 3.7. *The next statements are equivalent to each other:*

(i) $Z = A_G^{-,\textcircled{\dagger}}$.
(ii) $ZA^{k+1} = GA^{k+1}$ *and* $\mathcal{R}((A^{\textcircled{\dagger}})^*) = \mathcal{R}(Z^*)$.
(iii) $\mathcal{R}((A^{\textcircled{\dagger}})^*) = \mathcal{R}(Z^*)$, $\mathcal{N}(Z^*) = \mathcal{N}((GA^{k+1})^*)$ *and* $A^{\textcircled{\dagger}}AZ = A^{\textcircled{\dagger}}$.
(iv) $A^{\textcircled{\dagger}}AZ = A^{\textcircled{\dagger}}$, $\mathcal{R}(Z^*) = \mathcal{R}((A^{\textcircled{\dagger}})^*)$ *and* $ZA^{k+1} = GA^{k+1}$.
(v) $ZA^{k+1} = GA^{k+1}$ *and* $\mathcal{N}(Z) = \mathcal{N}(A^{\textcircled{\dagger}})$.

Proof. (i)⇒(ii) Let $Z = A_G^{-,\textcircled{\dagger}}$. Then $ZA^{k+1} = GAA^{\textcircled{\dagger}}A^{k+1} = GA^{k+1}$. Further,

$$Z^* = (A_G^{-,\textcircled{\dagger}})^* = (GAA^{\textcircled{\dagger}})^* = (A^{\textcircled{\dagger}})^*(GA)^*, \tag{3.7}$$

and

$$(A^{\textcircled{\dagger}})^* = (A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}})^* = (A^{\textcircled{\dagger}}AGAA^{\textcircled{\dagger}})^* = (A^{\textcircled{\dagger}}AZ)^* = (Z)^*(A^{\textcircled{\dagger}}A)^*. \tag{3.8}$$

From equations (3.7) and (3.8), we obtain $\mathcal{R}((A^{\textcircled{\dagger}})^*) = \mathcal{R}(Z^*)$.
(ii)⇒(i) Let $\mathcal{R}(Z^*) = \mathcal{R}((A^{\textcircled{\dagger}})^*)$. Then $Z = WA^{\textcircled{\dagger}}$ for some $W \in \mathbb{C}^{n\times n}$. Now,

$$\begin{aligned} Z &= WA^{\textcircled{\dagger}} = ZA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = ZAA^{\textcircled{\dagger}} = ZA^{k+1}(A^{\textcircled{\dagger}})^{k+1} = GA^{k+1}(A^{\textcircled{\dagger}})^{k+1} = GAA^{\textcircled{\dagger}} \\ &= A_G^{-,\textcircled{\dagger}}. \end{aligned}$$

(i)⇒(iii) Let $Z = A_G^{-,\textcircled{\dagger}}$. Then by the proof of (i)⇒(ii), we conclude $\mathcal{R}(Z^*) = \mathcal{R}((A^{\textcircled{\dagger}})^*)$. From the expressions

$$Z^* = (GAA^{\textcircled{\dagger}})^* = (GA^{k+1}(A^{\textcircled{\dagger}})^{k+1})^* = ((A^{\textcircled{\dagger}})^{k+1})^*(GA^{k+1})^*,$$

and

$$(GA^{k+1})^* = (GAA^{\textcircled{\dagger}}A^{k+1})^* = (A^{k+1})^*Z^*,$$

it follows $\mathcal{N}(Z^*) = \mathcal{N}((GA^{k+1})^*)$. Finally, $A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AGAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AZ$.
(iii)⇒(ii) It is enough to show only $ZA^{k+1} = GA^{k+1}$. Let $\mathcal{R}(Z^*) = \mathcal{R}((A^{\textcircled{\dagger}})^*)$. Then $Z = WA^{\textcircled{\dagger}}$ for some $W \in \mathbb{C}^{n\times n}$, and

$$Z = WA^{\textcircled{\dagger}} = (WA^{\textcircled{\dagger}})AA^{\textcircled{\dagger}} = ZAA^{\textcircled{\dagger}} = ZAA^{\textcircled{\dagger}}AZ.$$

Equivalently, $Z^*(I_n - ZAA^{\textcircled{\dagger}}A)^* = 0$. Thus, we obtain $\mathcal{R}(I_n - ZAA^{\textcircled{\dagger}}A)^* \subseteq \mathcal{N}(Z^*) = \mathcal{N}((GA^{k+1})^*)$ and consequently, $(I_n - ZAA^{\textcircled{\dagger}}A)(GA^{k+1}) = 0$. Hence,

$$GA^{k+1} = ZAA^{\textcircled{\dagger}}AGA^{k+1} = ZAA^{\textcircled{\dagger}}AGAA^k = ZAA^{\textcircled{\dagger}}A^{k+1} = ZA^{k+1}.$$

(i)⇒(iv) It is a consequence of $A^{\textcircled{\dagger}}AZ = A^{\textcircled{\dagger}}AA_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.
(iv)⇒(ii) Verification of this implication is trivial.
(ii)⇒(v) It follows from $\mathcal{R}(Z)^{\perp} = \mathcal{N}(Z^*)$. □

Theorem 3.8. *The subsequent characterizations of the* ICEP *inverse are equivalent:*

(i) $Z = A_G^{-,\textcircled{\dagger}}$.
(ii) $ZA = GAA^{\textcircled{\dagger}}A$ *and* $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}(Z)$.
(iii) $AZA = AA^{\textcircled{\dagger}}A$, $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}(Z)$ *and* $\mathcal{R}(GA^{k+1}) = \mathcal{R}(Z)$.
(iv) $GAZ = Z$, $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}(Z)$. *and* $ZA^{k+1} = GA^{k+1}$.

Proof. (i)⇒(ii) Let $Z = A_G^{-,\textcircled{\dagger}}$. Then $ZA = A_G^{-,\textcircled{\dagger}}A = GAA^{\textcircled{\dagger}}A$ and by Theorem 3.7, we get $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}(Z)$.
(ii)⇒(i) Let $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}(Z)$. Then $Z = WA^{\textcircled{\dagger}}$ for some $W \in \mathbb{C}^{n\times n}$. From $ZA = GAA^{\textcircled{\dagger}}A$, we obtain

$$Z = WA^{\textcircled{\dagger}} = WA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = (ZA)A^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}.$$

(ii)⇒(iii) Let $ZA = GAA^{\textcircled{\dagger}}A$. Then $AZA = AGAA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$. Using the equivalent conditions of (ii) and (i), we get $Z = GAA^{\textcircled{\dagger}} = GA^{k+1}(A^{\textcircled{\dagger}})^{k+1}$. Thus $\mathcal{R}(Z) \subseteq \mathcal{R}(GA^{k+1})$.
(iii)⇒(ii) Let $\mathcal{R}(Z) \subseteq \mathcal{R}(GA^{k+1})$. Then $Z = GA^{k+1}W$ for some $W \in \mathbb{C}^{n\times n}$. Further, from $GA^{k+1} = GAA^k = G(AGA)A^k = GAGA^{k+1}$, we get

$$ZA = GA^{k+1}WA = GAGA^{k+1}WA = GAZA = GAA^{\textcircled{\dagger}}A.$$

(i)⇒(iv) It follows from $ZA^{k+1} = GAA^{\textcircled{\dagger}}A^{k+1} = GA^{k+1}$ and

$$GAZ = GAA_G^{-,\textcircled{\dagger}} = G(AGA)A^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = Z.$$

(iv)⇒(i) Let $\mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}(Z)$. Then $Z = WA^{\textcircled{\dagger}}$ for some matrix $W \in \mathbb{C}^{n\times n}$. From $Z = GAZ$ and $ZA^{k+1} = GA^{k+1}$, we obtain

$$\begin{aligned} Z &= WA^{\textcircled{\dagger}} = WA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = ZAA^{\textcircled{\dagger}} = (GAZ)AA^{\textcircled{\dagger}} = GA(ZA^{k+1})(A^{\textcircled{\dagger}})^{k+1} \\ &= GAGA^{k+1}(A^{\textcircled{\dagger}})^{k+1} = GAGAA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} \end{aligned}$$

and finalize the proof. □

In the following theorem, we discuss the equivalency of the ICEP inverses with the core-EP inverse.

Theorem 3.9. *The subsequent representations hold:*

(i) $A_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}} \iff A^{\textcircled{\dagger}}A^k = GA^k \iff A_G^{-,\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$;

(ii) $AA_G^{-,\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A \Longrightarrow A_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.

(iii) $A_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}} \iff AA_G^{-,\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A^2A^{\textcircled{\dagger}}$.

Proof. (i) Let $A_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}}$. Then $(G - A^{\textcircled{\dagger}})AA^{\textcircled{\dagger}} = 0$. Thus $AA^{\textcircled{\dagger}}(G - A^{\textcircled{\dagger}})^* = 0$ and consequently,

$$\mathcal{R}(G - A^{\textcircled{\dagger}})^* \subseteq \mathcal{N}(AA^{\textcircled{\dagger}}) \subseteq \mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}((A^k)^*).$$

Hence $GA^k = A^{\textcircled{\dagger}}A^k$ and $A_G^{-,\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$ is trivial. Conversely, if $GA^k = A^{\textcircled{\dagger}}A^k$, then

$$A_G^{-,\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = GA^k(A^{\textcircled{\dagger}})^k = A^{\textcircled{\dagger}}A^k(A^{\textcircled{\dagger}})^k = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}.$$

Further, if $A_G^{-,\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$, then $A_G^{-,\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.
(ii) Let $AA_G^{-,\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A$. Then $AA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}A$ and

$$A^{\textcircled{\dagger}} = A(A^{\textcircled{\dagger}})^2 = GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}.$$

(iii) Let $A_G^{-,\textcircled{\dagger}} = A^{\textcircled{\dagger}}$. Then

$$AA_G^{-,\textcircled{\dagger}} = AA^{\textcircled{\dagger}} = A^k(A^{\textcircled{\dagger}})^k = A^{\textcircled{\dagger}}A^{k+1}(A^{\textcircled{\dagger}})^k = A^{\textcircled{\dagger}}A^2A^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A^2A^{\textcircled{\dagger}}.$$

Conversely, if $AA_G^{-,\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A^2A^{\textcircled{\dagger}}$. Then $AA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A^2A^{\textcircled{\dagger}}$. Post-multiplication by $A^{\textcircled{\dagger}}$ leads to $A^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}A^2(A^{\textcircled{\dagger}})^2 = A_G^{-,\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}$. □

The maximal classes for ICEP inverses are established in the subsequent two theorems.

Theorem 3.10. *Let $A \triangleright_n G$ and $W \in \mathbb{C}^{n\times n}$. Then the next statements are equivalent:*

(i) $A_G^{-,\textcircled{\dagger}} = GAW$.

(ii) $AW = AA^{\textcircled{\dagger}}$.

(iii) $AWA = AA^{\textcircled{\dagger}}A$ *and* $\mathcal{N}(AW) = \mathcal{N}(A^{\textcircled{\dagger}})$.

(iv) $W = A^{\textcircled{\dagger}} + (I_n - GA)Y$, *for arbitrary* $Y \in \mathbb{C}^{n\times n}$.

Proof. (i)⇒(ii) It follows from $AW = AGAW = AA_G^{-,\textcircled{\dagger}} = AA^{\textcircled{\dagger}}$.
(ii)⇒(iii) Clearly, $AWA = AA^{\textcircled{\dagger}}A$ and the null condition $\mathcal{N}(AW) = \mathcal{N}(A^{\textcircled{\dagger}})$ is follows from

$$AW = AA^{\textcircled{\dagger}}, \text{ and } A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AW.$$

(iii)⇒(i) From $\mathcal{N}(AW) = \mathcal{N}(A^{\textcircled{\dagger}})$, we obtain $AW = XA^{\textcircled{\dagger}}$ for some $X \in \mathbb{C}^{n\times n}$. Now pre-multiplying $AW = XA^{\textcircled{\dagger}}$, by G we get

$$\begin{aligned} GAW &= GXA^{\textcircled{\dagger}} = GXA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = GAWAA^{\textcircled{\dagger}} = GAWA^{k+1}(A^{\textcircled{\dagger}})^{k+1} \\ &= G(AWA)A^k(A^{\textcircled{\dagger}})^{k+1} = GAA^{\textcircled{\dagger}}AA^k(A^{\textcircled{\dagger}})^{k+1} = GAA^{\textcircled{\dagger}}A^{k+1}(A^{\textcircled{\dagger}})^{k+1} \\ &= GAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}. \end{aligned}$$

(ii)⇒(iv) We can easily verify that the general solution to the homogeneous equation $AW = 0$ will be of the form $W = (I_n - GA)Y$, where $Y \in \mathbb{C}^{n\times n}$. Since $A^{\textcircled{\dagger}}$ is a particular solution to $AW = AA^{\textcircled{\dagger}}$, the general solution of $AW = AA^{\textcircled{\dagger}}$ is given by

$$W = A^{\textcircled{\dagger}} + (I_n - GA)Y, \text{ for some } Y \in \mathbb{C}^{n\times n}.$$

(iv)⇒(i) Let $W = A^{\textcircled{\dagger}} + (I_n - GA)Y$ for some $Y \in \mathbb{C}^{n\times n}$. Pre-multiplying by GA on both sides, we obtain

$$GAW = GAA^{\textcircled{\dagger}} + GAY - GAGAY = GAA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}.$$

□

Theorem 3.11. *Let $A \rhd_n G$ and $S \in \mathbb{C}^{n\times n}$. Then the following characterizations hold:*

(i) $A_G^{-,\textcircled{\dagger}} = SAA^{\textcircled{\dagger}}$ *if and only if* $S = G + W(I_n - AA^{\textcircled{\dagger}})$ *for some* $W \in \mathbb{C}^{n\times n}$*;*

(ii) $A_G^{-,\textcircled{\dagger}} = SAT$ *if and only if* $S = G+W(I_n - AA^{\textcircled{\dagger}})$ *and* $T = A^{\textcircled{\dagger}} + (I_n - GA)X$ *for some* $T, X \in \mathbb{C}^{n\times n}$*.*

Proof. (i) The assumption $S = G+W(I_n - AA^{\textcircled{\dagger}})$ for some $W \in \mathbb{C}^{n\times n}$ implies $SAA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}$. Conversely, let $A_G^{-,\textcircled{\dagger}} = SAA^{\textcircled{\dagger}}$. Clearly, G is a particular solution of $SAA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}$. If there exists any other solution W of the homogeneous equation $SAA^{\textcircled{\dagger}} = 0$, then $WAA^{\textcircled{\dagger}} = 0$. Now we can write $W = W - WAA^{\textcircled{\dagger}} = W(I_n - AA^{\textcircled{\dagger}})$. Therefore, the general solution to homogeneous equation $SAA^{\textcircled{\dagger}} = 0$ is given by $S = W(I_n - AA^{\textcircled{\dagger}})$ and consequently, the general solution $SAA^{\textcircled{\dagger}} = A_G^{-,\textcircled{\dagger}}$ is expressed by

$$S = G + W(I_n - AA^{\textcircled{\dagger}}) \text{ for some } W \in \mathbb{C}^{n\times n}.$$

(ii) This follows directly from the part (i) and (iv) of Theorem 3.10. □

3.1.1. Computing ICEP inverse by the HS-decomposition

Now we present the canonical representation of ICEP inverses by considering the Hartwig and Spindelböck decomposition (in short HS-decomposition)(see [15, Corollary 6]). If $A \in \mathbb{C}^{n\times n}$ is any matrix having rank r, we can write the HS-decomposition of A as

$$A = V \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} V^*, \tag{3.9}$$

where $V \in \mathbb{C}^{n\times n}$ is any unitary matrix, the diagonal matrix $\Sigma = \operatorname{diag}(\sigma I_{r_1}, \sigma I_{r_2}, \ldots, \sigma I_{r_s})$ is the singular values of A such that $\sigma_1 > \cdots > \sigma_s > 0$, $r_1 + \cdots + r_s = r$ and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

If A is of the form (3.9), then we can calculate G i.e., an inner inverse of A as given below:

$$G = V \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} V^*,$$

where $\Sigma K Z_1 + \Sigma L Z_3 = I_r$. From[11] the core-EP inverse of A is of the form

$$A^{\textcircled{\dagger}} = V \begin{bmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{bmatrix} V^*. \tag{3.10}$$

Now, the ICEP inverses of A is given by

$$\begin{aligned} A_G^{-,\textcircled{\dagger}} = GAA^{\textcircled{\dagger}} &= V \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= V \begin{bmatrix} Z_1 \Sigma K (\Sigma K)^{\textcircled{\dagger}} & 0 \\ Z_3 \Sigma K (\Sigma K)^{\textcircled{\dagger}} & 0 \end{bmatrix} V^*. \end{aligned}$$

In view of the above calculations, we propose Theorem 3.12 as a conclusion.

Theorem 3.12. *Consider $A \in \mathbb{C}^{n\times n}$ as defined in* (3.9)*. In this case it follows*

$$A_G^{-,\textcircled{\dagger}} = V \begin{bmatrix} Z_1 \Sigma K (\Sigma K)^{\textcircled{\dagger}} & 0 \\ Z_3 \Sigma K (\Sigma K)^{\textcircled{\dagger}} & 0 \end{bmatrix} V^*,$$

where $\Sigma K Z_1 + \Sigma L Z_3 = I_r$.

3.2. A relation defined on the class of matrices using ICEP inverse

In this subsection we introduce a binary relation based on ICEP inverses.

Definition 3.2. *Let $A \triangleright_n G$ and $B \in \mathbb{C}^{n\times n}$. We say A is below B under the binary relation $\leq^{-,\textcircled{\dagger}}$ and write $A \leq^{-,\textcircled{\dagger}} B$ if $A_G^{-,\textcircled{\dagger}} A = A_G^{-,\textcircled{\dagger}} B$ and $AA_G^{-,\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}$.*

Proposition 3.1. *Assume $A \triangleright_n G$ and $B \in \mathbb{C}^{n\times n}$. Then*

(i) $A_G^{-,\textcircled{\dagger}} A = A_G^{-,\textcircled{\dagger}} B \iff A^{\textcircled{\dagger}} A = A^{\textcircled{\dagger}} B$;

(ii) $AA^{\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}} \iff A^{\textcircled{\dagger}} = BGA^{\textcircled{\dagger}}$.

Proof. (i) Let $A_G^{-,\textcircled{\dagger}} A = A_G^{-,\textcircled{\dagger}} B$. Then

$$AA^{\textcircled{\dagger}}A = AGAA^{\textcircled{\dagger}}A = AA_G^{-,\textcircled{\dagger}} A = AA_G^{-,\textcircled{\dagger}} B = AA^{\textcircled{\dagger}}B. \tag{3.11}$$

Pre-multiplying equation (3.11) by $A^{\textcircled{\dagger}}$, we conclude $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$. The converse part is trivial.
(ii) Let $AA_G^{-,\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}$. Then $AA^{\textcircled{\dagger}} = BGAA^{\textcircled{\dagger}}$. Now

$$A^{\textcircled{\dagger}} = (AA^{\textcircled{\dagger}})A^{\textcircled{\dagger}} = BG(AA^{\textcircled{\dagger}}A^{\textcircled{\dagger}}) = BGA^{\textcircled{\dagger}}.$$

Conversely, $A^{\textcircled{\dagger}} = BGA^{\textcircled{\dagger}}$ leads to

$$\begin{aligned} AA^{\textcircled{\dagger}} &= A^k(A^{\textcircled{\dagger}})^k = A^{\textcircled{\dagger}}A^{k+1}(A^{\textcircled{\dagger}})^k \\ &= BGA^{\textcircled{\dagger}}A^{k+1}(A^{\textcircled{\dagger}})^k = BGA^k(A^{\textcircled{\dagger}})^k \\ &= BGAA^{\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}, \end{aligned}$$

which was our intention. □

Corollary 3.3. *Let $A \triangleright_n G$ and $B \in \mathbb{C}^{n\times n}$. Then the subsequent statements are equivalent:*

(i) $A \leqslant^{-,\textcircled{\dagger}} B$.

(ii) $AA^{\textcircled{\dagger}}A = BA_G^{-,\textcircled{\dagger}} A = AA^{\textcircled{\dagger}}B$.

(iii) $AA^{\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}$ *and* $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$.

Proof. (i)⇒(ii) Let $A \leqslant^{-,\textcircled{\dagger}} B$. Then $AA^{\textcircled{\dagger}}A = AGAA^{\textcircled{\dagger}}A = (AA_G^{-,\textcircled{\dagger}})A = BA_G^{-,\textcircled{\dagger}} A$ and

$$AA^{\textcircled{\dagger}}A = AGAA^{\textcircled{\dagger}}A = A(A_G^{-,\textcircled{\dagger}} A) = AA_G^{-,\textcircled{\dagger}} B = AA^{\textcircled{\dagger}}B.$$

(ii)⇒(iii) Let $AA^{\textcircled{\dagger}}A = BA_G^{-,\textcircled{\dagger}} A = AA^{\textcircled{\dagger}}B$. Then

$$A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}B = A^{\textcircled{\dagger}}B, \text{ and}$$

$$AA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}} AA^{\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}.$$

(iii)⇒(i) This implication follows directly from $AA_G^{-,\textcircled{\dagger}} = AA^{\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}$ and Proposition 3.1(i). □

Notice that the binary relation $\leqslant^{-,\textcircled{\dagger}}$ is reflexive. However, it is neither symmetric nor anti-symmetric, as shown below.

Example 3.5. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Selected inner inverses of A and B are given by

$$G_A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \text{ and } G_B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In addition, $A^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} = A^{-,\textcircled{\dagger}}_{G_A} = B^{-,\textcircled{\dagger}}_{G_B} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Further, we evaluate,

$AA^{-,\textcircled{\dagger}}_{G_A} = BA^{-,\textcircled{\dagger}}_{G_A} = AB^{-,\textcircled{\dagger}}_{G_B} = BB^{-,\textcircled{\dagger}}_{G_B} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and

$A^{-,\textcircled{\dagger}}_{G_A} A = A^{-,\textcircled{\dagger}}_{G_A} B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B^{-,\textcircled{\dagger}}_{G_B} A = B^{-,\textcircled{\dagger}}_{G_B} B.$

Thus, $A \leq^{-,\textcircled{\dagger}} B$ and $B \leq^{-,\textcircled{\dagger}} A$ but $A \neq B$. Hence, the relation $\leq^{-,\textcircled{\dagger}}$ is not anti-symmetric.

Now for a matrix $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ we choose $G_C = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ and compute

$$C^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} \\ 0 & \frac{4}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}, \quad C^{-,\textcircled{\dagger}}_{G_C} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

Obtained results lead to the conclusion $A \leq^{-,\textcircled{\dagger}} C$ but $C \not\leq^{-,\textcircled{\dagger}} A$, since $C^{-,\textcircled{\dagger}}_{G_C} A \neq C^{-,\textcircled{\dagger}}_{G_C} C$. Hence, the relation $\leq^{-,\textcircled{\dagger}}$ is not symmetric.

The following example shows that the binary relation $\leq^{-,\textcircled{\dagger}}$ is not transitive.

Example 3.6. Consider input matrices $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Let us choose inner inverses of A and B, respectively, as

$$G_A = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix} \text{ and } G_B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and evaluate $A^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Further calculation gives $AA^{-,\textcircled{\dagger}}_{G_A} = BA^{-,\textcircled{\dagger}}_{G_A}$, $B^{-,\textcircled{\dagger}}_{G_B} B = B^{-,\textcircled{\dagger}}_{G_B} C$ and $BB^{-,\textcircled{\dagger}}_{G_B} = CB^{-,\textcircled{\dagger}}_{G_B}$. Thus

$$A^{-,\textcircled{\dagger}}_{G_A} A = A^{-,\textcircled{\dagger}}_{G_A} C \text{ but } AA^{-,\textcircled{\dagger}}_{G_A} = A^{\textcircled{\dagger}} \neq \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix} = CA^{-,\textcircled{\dagger}}_{G_A}.$$

Since $A \leq^{-,\textcircled{\dagger}} B$ and $B \leq^{-,\textcircled{\dagger}} C$ but $A \not\leq^{-,\textcircled{\dagger}} C$, the relation $\leq^{-,\textcircled{\dagger}}$ is not transitive.

The next theorem will provide all elements B such as $A \leq^{-,\textcircled{\dagger}} B$ for a given matrix A.

Theorem 3.13. *Consider $A \in \mathbb{C}^{n\times n}$ as defined in* (3.2) *and let $G \in A\{1\}$ be as in* (3.3). *If $B \in \mathbb{C}^{n\times n}$ then the subsequent statements are equivalent:*

(i) $A \leq^{-,\textcircled{\dagger}} B$;

(ii) $B = U\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}U^*$, *where* $B_1Z_1 + B_2Z_3 = I_r$, $B_2Z_1 + B_4Z_3 = 0$, $Z_1T_1 = Z_1B_1$, $Z_3T_1 = Z_3B_1$, $Z_1T_2 = Z_1B_2$, *and* $Z_3T_2 = Z_3B_2$.

Proof. (i)⇒(ii) Let $A \leq^{-,\textcircled{\dagger}} B$ holds for any $B \in \mathbb{C}^{n\times n}$. From Theorem 3.2, we can write $A_G^{-,\textcircled{\dagger}} = U\begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix}U^*$, for some $Z_1 \in \mathbb{C}^{r\times r}$, and $Z_3 \in \mathbb{C}^{(n-r)\times r}$ with $T_1Z_1 + T_2Z_3 = I_r$, $NZ_3 = 0$. Let $B = U\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}U^*$ be partitioned as the sizes of the blocks of A. From $A_G^{-,\textcircled{\dagger}} A = A_G^{-,\textcircled{\dagger}} B$, we can derive that

$$Z_1T_1 = Z_1B_1,\ Z_3T_1 = Z_3B_1,\ Z_1T_2 = Z_1B_2, \text{ and } Z_3T_2 = Z_3B_2.$$

Similarly, $AA_G^{-,\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}$ gives $B_1Z_1 + B_2Z_3 = I_r,\ B_2Z_1 + B_4Z_3 = 0$.
(ii)⇒(i) It is a straightforward verification. □

Under a few suitable conditions, we discuss the transitive property of ICEP inverses, in Theorem 3.14.

Theorem 3.14. *Let* $A \triangleright_n G$ *and* $B \in \mathbb{C}^{n\times n}$ *be two matrices of same index. Suppose* $\|A^{\textcircled{\dagger}}\| \leq 1$ *and* $\lim\limits_{k\to\infty} \|A^kA^{\textcircled{\dagger}} - GA^k\| = 0$. *Under these conditions* $A \leq^{-,\textcircled{\dagger}} B \iff A \leq^{\textcircled{\dagger}} B$.

Proof. Let $A \leq^{-,\textcircled{\dagger}} B$. Then from $A_G^{-,\textcircled{\dagger}} A = A_G^{-,\textcircled{\dagger}} B$, we get

$$A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AGAA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AA_G^{-,\textcircled{\dagger}} A = A^{\textcircled{\dagger}}AA_G^{-,\textcircled{\dagger}} B = A^{\textcircled{\dagger}}B.$$

Similarly, from $AA^{\textcircled{\dagger}} = AA_G^{-,\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}} = BGAA^{\textcircled{\dagger}}$, we have

$$\begin{aligned} AA^{\textcircled{\dagger}} - BA^{\textcircled{\dagger}} &= BGAA^{\textcircled{\dagger}} - BA^{\textcircled{\dagger}} = BGA^k(A^{\textcircled{\dagger}})^k - BA(A^{\textcircled{\dagger}})^2 \\ &= BGA^k(A^{\textcircled{\dagger}})^k - BA^k(A^{\textcircled{\dagger}})^kA^{\textcircled{\dagger}} = (BGA^k - BA^kA^{\textcircled{\dagger}})(A^{\textcircled{\dagger}})^k \\ &= B(GA^k - A^kA^{\textcircled{\dagger}})(A^{\textcircled{\dagger}})^k. \end{aligned}$$

Therefore, $\|AA^{\textcircled{\dagger}} - BA^{\textcircled{\dagger}}\| \leq \|B\|\|GA^k - A^kA^{\textcircled{\dagger}}\|\|(A^{\textcircled{\dagger}})^k\| \to 0$ as $k \to \infty$ and hence $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.

Conversely, let $A \leq^{\textcircled{\dagger}} B$. Then $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$. Further,

$$A_G^{-,\textcircled{\dagger}} A = GAA^{\textcircled{\dagger}}A = GAA^{\textcircled{\dagger}}B = A_G^{-,\textcircled{\dagger}} B$$

and

$$\begin{aligned} AA_G^{-,\textcircled{\dagger}} - BA_G^{-,\textcircled{\dagger}} &= AA^{\textcircled{\dagger}} - BGAA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}} - BGAA^{\textcircled{\dagger}} \\ &= BA(A^{\textcircled{\dagger}})^2 - BGA^k(A^{\textcircled{\dagger}})^k = BA^k(A^{\textcircled{\dagger}})^{k+1} - BGA^k(A^{\textcircled{\dagger}})^k \\ &= B(A^kA^{\textcircled{\dagger}} - GA^k)(A^{\textcircled{\dagger}})^k. \end{aligned}$$

Therefore, $\|AA_G^{-,\textcircled{\dagger}} - BA_G^{-,\textcircled{\dagger}}\| \leq \|B\|\|A^kA^{\textcircled{\dagger}} - GA^k\|\|(A^{\textcircled{\dagger}})^k\| \to 0$ as $k \to \infty$. Hence $AA_G^{-,\textcircled{\dagger}} = BA_G^{-,\textcircled{\dagger}}$ and completes the proof. □

Remark 3.2. *If* $A,\ B \in \mathcal{PO}$ *(where* $\mathcal{PO} = \{A \in \mathbb{C}^{n\times n} : \|A^{\textcircled{\dagger}}\| \leq 1$ *and* $\lim\limits_{k\to\infty} \|A^kA^{\textcircled{\dagger}} - GA^k\| = 0\}$, *the relation* $\leq^{-,\textcircled{\dagger}}$ *represents a pre-order on* $\mathcal{PO}$.

4. CEPI inverse

This section contains a few representations of CEPI (or core-EP inner) inverse for square matrices. Since the proofs are identical to the proofs of ICEP inverses, we will exclude the proofs.

Theorem 4.1. *Let $A \triangleright_n G$. Then $Z = A^{\textcircled{\dagger}}AG$ is the unique solver of the matrix equations*

$$ZAZ = Z, \ ZA = A^{\textcircled{\dagger}}A, \ \text{and } AZ = AA^{\textcircled{\dagger}}AG.$$

Considering Theorem 3.1, we define CEPI inverses for square matrices in Definition 4.1.

Definition 4.1. *$A \triangleright_n G$. The* CEPI *inverse of A is defined as*

$$A_G^{\textcircled{\dagger},-} = A^{\textcircled{\dagger}}AG.$$

The set of CEPI *inverses of A is defined by $A\{\textcircled{\dagger},-\} = \{A^{\textcircled{\dagger}}AG |\ G \in A\{1\}\}$.*

Example 4.1. Choose the matrices A and G. from Example 3.1. Exact calculation gives

$$A_G^{\textcircled{\dagger},-} = A^{\textcircled{\dagger}}AG = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is observable that $A_G^{\textcircled{\dagger},-} \notin \{A^{-,\mathrm{D}}, A^{\dagger,\textcircled{\dagger}}, A^{\dagger,\mathrm{D}}, A^{\mathrm{D},\dagger}, A^{c,\dagger}, A^{\textcircled{\dagger},\dagger}, A_G^{-,\textcircled{\dagger}}\}$.
Further,

$$A\{\textcircled{\dagger},-\} = A^{\textcircled{\dagger}}AA\{1\} = \left\{ \begin{bmatrix} 1 & 0 & x_{1,3}+x_{4,3} & x_{1,4}+x_{4,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid x_{ij} \in \mathbb{C} \right\}.$$

Clearly, $A_G^{\textcircled{\dagger},-}$ is derived in the particular case $x_{13}+x_{43} = 9$, $x_{14}+x_{44} = 0$ in $A\{\textcircled{\dagger},-\}$.

Similarly, the following results can be derived in the case of the dual (CEPI inverses).

Proposition 4.1. *Let $A \in \mathbb{C}^{n\times n}$ and $l \geqslant k$ be any non-negative integer. Then*

(i) $A_G^{\textcircled{\dagger},-} = A^{\mathrm{D}}A^l(A^l)^{\dagger}AG = A^{\mathrm{D}}P_{\mathcal{R}(A^l)}AG = A^l\left(A^{l+1}\right)^{\dagger}AG$;
(ii) $A_G^{\textcircled{\dagger},-}A^{l+1} = A^l$;
(iii) $A^{l+1}A_G^{\textcircled{\dagger},-} = A^{2l}(A^l)^{\dagger}AG$.

Lemma 4.1. *Let $A \in \mathbb{C}^{n\times n}$. Then*

(i) *$A_G^{\textcircled{\dagger},-}A$ is an orthogonal projector onto $\mathcal{R}(A^{\textcircled{\dagger}}A)$;*
(ii) *$AA_G^{\textcircled{\dagger},-}$ is a projector onto $\mathcal{R}(AA^{\textcircled{\dagger}})$ along $\mathcal{N}((A^k)^{\dagger}AG)$.*

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$. Suppose $P' = I_n - A_G^{\textcircled{\dagger},-}A$ and $P'' = I_n - AA_G^{\textcircled{\dagger},-}$. Then, for $A+P''$ and $A-P''$ both invertible,*

$$A_G^{\textcircled{\dagger},-} = (I_n - P')(A \pm P'')^{-1}(I_n - P'').$$

Theorem 4.3. *The upcoming characterizations are equivalent in the case $A \triangleright_n G$:*

(i) $Z = A_G^{\textcircled{\dagger},-}$.
(ii) $ZAZ = Z,\ AZ = AA^{\textcircled{\dagger}}AG,\ ZA = A^{\textcircled{\dagger}}A$ *and* $AZA = AA^{\textcircled{\dagger}}A$.
(iii) $ZA = A^{\textcircled{\dagger}}A$ *and* $ZAA^{\textcircled{\dagger}}AG = Z$.
(iv) $AZ = AA^{\textcircled{\dagger}}AG$ *and* $A^{\textcircled{\dagger}}AZ = Z$.

(v) $ZAA^{\textcircled{\dagger}}AZ = Z$, $ZAA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$, $AA^{\textcircled{\dagger}}AZ = AA^{\textcircled{\dagger}}AG$ *and* $AA^{\textcircled{\dagger}}AZAA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$.
(vi) $ZAA^{\textcircled{\dagger}}AZ = Z$, $ZAA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$ *and* $AA^{\textcircled{\dagger}}AZ = AA^{\textcircled{\dagger}}AG$.

Theorem 4.4. *Let $A \triangleright_n G$. Then the next characterizations hold:*
(i) $AA_G^{\textcircled{\dagger},-} = AG \Longleftrightarrow AA^{\textcircled{\dagger}}A = A$;
(ii) $AA_G^{\textcircled{\dagger},-} = AA^{\textcircled{\dagger}} \Longleftrightarrow A_G^{\textcircled{\dagger},-} = A^{\textcircled{\dagger}}$;
(iii) $A_G^{\textcircled{\dagger},-} = (A^{\textcircled{\dagger}})^2 A \Longleftrightarrow AA_G^{\textcircled{\dagger},-} = A_G^{\textcircled{\dagger},-}A$.

Theorem 4.5. *If $A \triangleright_n G$ and $W \in \mathbb{C}^{n\times n}$, the following characterizations are equivalent:*
(i) $A_G^{\textcircled{\dagger},-} = WAG$.
(ii) $WA = A^{\textcircled{\dagger}}A$.
(iii) $AWA = AA^{\textcircled{\dagger}}A$ *and* $\mathcal{R}(WA) = \mathcal{R}(A^{\textcircled{\dagger}})$.
(iv) $W = A^{\textcircled{\dagger}} + Y(I_n - AG)$, *for any* $Y \in \mathbb{C}^{n\times n}$.

Example 4.2. Consider the input matrix of index $\text{ind}(A) = 4$:

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The set of inner inverses is defined by

$$A\{1\} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} \\ 0 & 2 & 0 & 0 & -2 & 0 & 0 & x_{2,8} \\ 0 & 0 & 2 & 0 & 0 & -2 & 0 & x_{3,8} \\ 0 & 0 & 0 & 2 & 0 & 0 & -2 & x_{4,8} \\ 2-x_{1,1} & -x_{1,2} & -x_{1,3} & -x_{1,4} & -x_{1,5} & -x_{1,6} & -x_{1,7} & x_{5,8} \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & x_{6,8} \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & x_{7,8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & x_{8,8} \end{bmatrix} \mid x_{ij} \in \mathbb{C} \right\}.$$

Then

$$A^{\textcircled{\dagger}} = A^4 \left(A^5\right)^\dagger = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which gives

$$A\{\textcircled{\dagger},-\} = A^{\textcircled{\dagger}}AA\{1\} = \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2-x_{1,1} & -x_{1,2} & -x_{1,3} & -x_{1,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid x_{1j} \in \mathbb{C} \right\}.$$

Theorem 4.6. *Let $A \rhd_n G$. Consider the decomposition's of A, G, and $A^{\textcircled{\dagger}}$ respectively as defined in* (3.2), (3.3), *and* (3.4). *Then* CEPI *inverses of A are represented as*

$$A_G^{\textcircled{\dagger},-} = U \begin{bmatrix} T_1^{-1} & Z_2 + T_1^{-1}T_2Z_4 \\ 0 & 0 \end{bmatrix} U^*,$$

where $(T_1Z_2 + T_2Z_4)N = 0$ and $Z_4 \in N\{1\}$.

Theorem 4.7. *Consider $A \in \mathbb{C}^{n\times n}$ as defined in* (3.9). *Then*

$$A_G^{\textcircled{\dagger},-} = V \begin{bmatrix} (\Sigma K)^{\textcircled{\dagger}} & (\Sigma K)^{\textcircled{\dagger}}(\Sigma K Z_2 + \Sigma L Z_4) \\ 0 & 0 \end{bmatrix} V^*,$$

where $\Sigma K Z_1 + \Sigma L Z_3 = I_r$.

Definition 4.2 proposes a binary relation for CEPI inverses.

Definition 4.2. *Let $A \rhd_n G$ and $B \in \mathbb{C}^{n\times n}$. We say A is below B under the binary relation $\leq^{\textcircled{\dagger},-}$ if $A_G^{\textcircled{\dagger},-}A = A_G^{\textcircled{\dagger},-}B$ and $AA_G^{\textcircled{\dagger},-} = BA_G^{\textcircled{\dagger},-}$. Such relation is denoted by $A \leq^{\textcircled{\dagger},-} B$.*

Proposition 4.2. *The following representations are equivalent for $A \rhd_n G$:*

(i) $A \leqslant^{\textcircled{\dagger},-} B$.

(i) $AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}A = AA_G^{\textcircled{\dagger},-}B$.

(iii) $A^{\textcircled{\dagger}}A = A_G^{\textcircled{\dagger},-}B$ *and* $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.

Theorem 4.8. *Let $A \rhd_n G$. Observe the decompositions of A, G, and $A^{\textcircled{\dagger}}$ as in* (3.2), (3.3), *and* (3.4), *respectively. If $B \in \mathbb{C}^{n\times n}$ then the next statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger},-} B$.

(ii) $B = U \begin{bmatrix} T_1 & B_2 \\ 0 & B_4 \end{bmatrix} U^*$, *where $B_2 = T_2 - (T_1Z_2 + T_2Z_4)B_4$ and $Z_4 \in N\{1\}$.*

Theorem 4.9. *Let $A \rhd_n G$ and $B \in \mathbb{C}^{n\times n}$ be two matrices of identical index. Suppose $\|A^{\textcircled{\dagger}}\|$ is bounded and $AG = AA^{\textcircled{\dagger}}$. In this case $A \leq^{\textcircled{\dagger},-} B \iff A \leq^{\textcircled{\dagger}} B$.*

5. P-core-EP inverse

In this section, we introduce the notion of P-core inverse on the set of square matrices of arbitrary index. First, we define the P-core-EP inverse of a square matrices as given below.

Definition 5.1. *Let $A \in \mathbb{C}^{n\times n}$. A matrix Z is called P-core-EP inverse of A if it satisfies*

$$ZA = A^{\textcircled{\dagger}}A \text{ and } Z(A-I) = (A-I)A^{\textcircled{\dagger}}. \tag{5.1}$$

Theorem 5.1. *Arbitrary matrix $A \in \mathbb{C}^{n\times n}$ of the pattern* (3.2) *satisfies*

$$Z = U \begin{bmatrix} T_1^{-1} & T_1^{-1}T_2 \\ 0 & 0 \end{bmatrix} U^* \tag{5.2}$$

is the unique solver of the system (5.1).

Proof. Let

$$Z = U \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} U^*$$

be a matrix for suitable blocks, satisfying (5.1) and $A^{\textcircled{\dagger}}$ is written as (3.4). From $ZA = A^{\textcircled{\dagger}}A$, we derive

$$U \begin{bmatrix} Z_1T_1 & Z_1T_2 + Z_2N \\ Z_3T_1 & Z_3T_2 + Z_4N \end{bmatrix} U^* = U \begin{bmatrix} I & T_1^{-1}T_2 \\ 0 & 0 \end{bmatrix} U^*.$$

Then, after some calculations, we obtain $Z_1 = T_1^{-1}, Z_3 = 0$. Further, $Z(A-I) = (A-I)A^{\textcircled{\dagger}}$ implies

$$U \begin{bmatrix} Z_1T_1 - Z_1 & Z_1T_2 - Z_2 \\ Z_3T_1 - Z_3 & Z_3T_2 - Z_4 \end{bmatrix} U^* = U \begin{bmatrix} I - T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Now by putting $Z_1 = T_1^{-1}$ and $Z_3 = 0$ above, we obtain $Z_2 = T_1^{-1}T_2$ and $Z_4 = 0$. Therefore,

$$Z = U \begin{bmatrix} T_1^{-1} & T_1^{-1}T_2 \\ 0 & 0 \end{bmatrix} U^*.$$

To show uniqueness, we suppose Z_1 and Z_2 are two solutions of the system (5.1). From first part of (5.1), we get

$$\begin{cases} Z_1A = A^{\textcircled{\dagger}}A, \\ Z_2A = A^{\textcircled{\dagger}}A. \end{cases} \tag{5.3}$$

On solving these two equations, we get

$$(Z_1 - Z_2)A = 0. \tag{5.4}$$

Similarly, from second part of (5.1), we have

$$\begin{cases} Z_1(A-I) = (A-I)A^{\textcircled{\dagger}}, \\ Z_2(A-I) = (A-I)A^{\textcircled{\dagger}}. \end{cases} \tag{5.5}$$

Again, by solving these two equations, we get

$$(Z_1 - Z_2)A = Z_1 - Z_2. \tag{5.6}$$

From (5.4) and (5.6), we obtain $Z_1 = Z_2$. Hence, Z is the unique solution. □

Corollary 5.1 is derived from Definition 5.1.

Corollary 5.1. *For $A \in \mathbb{C}^{n\times n}$ satisfying $k = \operatorname{ind}(A)$, the unique solution to (5.1) is equal to*

$$Z = A^{\textcircled{\dagger}} + A^{\textcircled{\dagger}}A - AA^{\textcircled{\dagger}}$$

and is denoted by $A^{\textcircled{\dagger}_p}$.

Proof. The proof is follows from (5.1) by simple calculation. □

Next, we present some properties of P-core-EP inverse.

Theorem 5.2. *For $A \in \mathbb{C}^{n\times n}$ satisfying $\operatorname{ind}(A) = k$ it follows*

(i) $A^{\textcircled{\dagger}_p}A^k = A^{\textcircled{\dagger}}A^k$,
(ii) $A^{\textcircled{\dagger}_p}A^k(A^k)^\dagger = A^{\textcircled{\dagger}}$,
(iii) $(A^{\textcircled{\dagger}_p})^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^{\textcircled{\dagger}_p}$.

Proof. (i) Let A be of the form (3.2). By substituting equation (5.2) and (3.10) into $A^{\textcircled{\dagger}_p}$ and $A^{\textcircled{\dagger}}$, we obtain

$$A^{\textcircled{\dagger}_p}A^k = U \begin{bmatrix} T_1^{k-1} & T_1^{-1}T \\ 0 & 0 \end{bmatrix} U^* = A^{\textcircled{\dagger}}A^k.$$

with $A^k = U \begin{bmatrix} T_1^k & T \\ 0 & 0 \end{bmatrix} U^*$ where $T = \sum_{i=0}^{k} T_1^i T_2 N^{k-i}$.
(ii) Similarly, from direct calculation, we obtain

$$A^{\textcircled{\dagger}_p} A^k (A^k)^\dagger = U \begin{bmatrix} T_1^{-1} & T_1^{-1}T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{rank(A^k)} & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = A^{\textcircled{\dagger}}.$$

(iii) It is easy to obtain

$$(A^{\textcircled{\dagger}_p})^{\textcircled{\dagger}} = U \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

According to Corollary5.1, we get

$$(A^{\textcircled{\dagger}})^{\textcircled{\dagger}_p} = (A^{\textcircled{\dagger}})^{\textcircled{\dagger}} + (A^{\textcircled{\dagger}})^{\textcircled{\dagger}} A^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} (A^{\textcircled{\dagger}})^{\textcircled{\dagger}}.$$

Now by using equation(3.2) and (3.4), we obtain

$$(A^{\textcircled{\dagger}})^{\textcircled{\dagger}_p} = U \left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \right) U^* = U \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Hence $(A^{\textcircled{\dagger}_p})^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^{\textcircled{\dagger}_p}$. □

Theorem 5.3. *If $A \in \mathbb{C}^{n\times n}$ is of index* $\text{ind}(A) = k$ *then* $A^{\textcircled{\dagger}_p} \in A\{1^k, 2, 6\}$.

Proof. From Theorem 5.2(ii), we have $A^{\textcircled{\dagger}_p} A^k = A^{\textcircled{\dagger}} A^k$. Now post-multiplying by A on both sides, we get

$$A^{\textcircled{\dagger}_p} A^{k+1} = A^{\textcircled{\dagger}} A^{k+1} = A^k.$$

Hence, $A^{\textcircled{\dagger}_p} \in A\{1^k\}$.
Suppose A is of the form (3.2). Then

$$A^{\textcircled{\dagger}_p} A A^{\textcircled{\dagger}_p} = U \begin{bmatrix} T_1^{-1} & T_1^{-1}T_2 \\ 0 & 0 \end{bmatrix} U^* = A^{\textcircled{\dagger}_p}.$$

Therefore, $A^{\textcircled{\dagger}_p} \in A\{2\}$. Also the identity $A(A^{\textcircled{\dagger}_p})^2 = A^{\textcircled{\dagger}_p}$ follows from direct computation. Therefore, $A^{\textcircled{\dagger}_p} \in A\{6\}$. □

Next, we present some relationships of P-core-EP inverse with other generalized inverse.

Theorem 5.4. *If $A \in \mathbb{C}^{n\times n}$ satisfies* $\text{rank}(A) = r$, *the subsequent equalities are equivalent:*

(i) $(AA^{\textcircled{\dagger}_p})^* = AA^{\textcircled{\dagger}_p}$,
(ii) $(A^{\textcircled{\dagger}} A)^* = A^{\textcircled{\dagger}} A$,
(iii) $A^{\textcircled{\dagger}_p} = A^{\textcircled{\dagger}}$.

Proof. Let A be of the form(3.2). Then by using equation (3.4) and (5.2), it is not difficult to conclude that all the three conditions are equivalent to $T_2 = 0$. □

5.1. Binary relation on the P-core-EP inverse

In view of the binary relation defined on P-core inverse, we now define the following relation for P-core-EP inverse.

Definition 5.2. *Let $A, B \in \mathbb{C}^{n\times n}$. It is said that A is below B under the relation $\leq^{\textcircled{\dagger}_p}$ if $A^{\textcircled{\dagger}_p} A = A^{\textcircled{\dagger}_p} B$ and $AA^{\textcircled{\dagger}_p} = BA^{\textcircled{\dagger}_p}$. We denote the relation by $A \leq^{\textcircled{\dagger}_p} B$.*

Theorem 5.5. *Let $A \in \mathbb{C}^{n\times n}$ as defined in* (3.2) *and $B \in \mathbb{C}^{n\times n}$, then the following are equivalent:*

(i) $A \leq^{\textcircled{\dagger}_p} B$;

(ii) $B = U \begin{bmatrix} T_1 & T_2 + T_2 N - T_2 B_4 \\ 0 & B_4 \end{bmatrix} U^*$, *where* $T_1 \in \mathbb{C}^{r\times r}$, $T_2 \in \mathbb{C}^{r\times n-r}$, *and* $B_4 \in \mathbb{C}^{n-r\times n-r}$ *of index* k.

Proof. (i)⇒(ii) Suppose $A \leq^{\textcircled{\dagger}_p} B$ satisfies for any $B \in \mathbb{C}^{n\times n}$. The equation (5.2) leads to the conclusion $A^{\textcircled{\dagger}_p} = U \begin{bmatrix} T_1^{-1} & T_1^{-1}T_2 \\ 0 & 0 \end{bmatrix} U^*$. Let $B = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^*$ be partitioned as the suitable block sizes of A. From $A^{\textcircled{\dagger}_p} A = A^{\textcircled{\dagger}_p} B$, we get

$$B_2 = T_2 + T_2 N - T_2 B_4.$$

Again, $AA^{\textcircled{\dagger}_p} = BA^{\textcircled{\dagger}_p}$ gives $B_1 = T_1$ and $B_3 = 0$.

(ii)⇒(i) It is obvious. □

From the definition, It is trivial that the binary relation $\leqslant^{\textcircled{\dagger}_p}$ is reflexive. However, it is neither symmetric nor anti-symmetric, as shown below.

Example 5.1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Further evaluation gives

$$A^{\textcircled{\dagger}_p} = B^{\textcircled{\dagger}_p} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Also, we evaluate,

$$AA^{\textcircled{\dagger}_p} = BA^{\textcircled{\dagger}_p} = A^{\textcircled{\dagger}_p}A = A^{\textcircled{\dagger}_p}B = AB^{\textcircled{\dagger}_p} = BB^{\textcircled{\dagger}_p} = B^{\textcircled{\dagger}_p}A = B^{\textcircled{\dagger}_p}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $A \leq^{\textcircled{\dagger}_p} B$ and $B \leq^{\textcircled{\dagger}_p} A$ but $A \neq B$. Hence, the relation $\leq^{\textcircled{\dagger}_p}$ is not anti-symmetric. Now consider a matrix $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ with $C^{\textcircled{\dagger}_p} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{4} & -\frac{3}{4} \end{bmatrix}$. Numerical experience shows $A \leq^{\textcircled{\dagger}_p} C$ but $C \not\leq^{\textcircled{\dagger}_p} A$, since $C^{\textcircled{\dagger}_p}A \neq C^{\textcircled{\dagger}_p}C$ and $AC^{\textcircled{\dagger}_p} \neq CC^{\textcircled{\dagger}_p}$. Hence, the relation $\leq^{\textcircled{\dagger}_p}$ is not symmetric.

Example 5.2 shows that the binary relation $\leq^{\textcircled{\dagger}_p}$ is not transitive.

Example 5.2. For input matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

simple calculation gives

$$A^{\textcircled{\dagger}_p} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B^{\textcircled{\dagger}_p} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Further calculation gives

$$AA^{\textcircled{\dagger}_p} = BA^{\textcircled{\dagger}_p},\ A^{\textcircled{\dagger}_p}A = A^{\textcircled{\dagger}_p}B,\ B^{\textcircled{\dagger}_p}B = B^{\textcircled{\dagger}_p}C \text{ and } BB^{\textcircled{\dagger}_p} = CB^{\textcircled{\dagger}_p}.$$

Thus

$$AA^{\textcircled{\dagger}_p} = CA^{\textcircled{\dagger}_p} \text{ but } A^{\textcircled{\dagger}_p}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = CA^{\textcircled{\dagger}_p}.$$

Since $A \leq^{\textcircled{\dagger}_p} B$ and $B \leq^{\textcircled{\dagger}_p} C$ but $A \not\leq^{\textcircled{\dagger}_p} C$, the relation $\leq^{\textcircled{\dagger}_p}$ is not transitive.

Remark 5.1. *From the above two examples we decide that the relation $\leq^{\textcircled{\dagger}}$ does not define a partial order. But if $\operatorname{ind}(A) = 1$, then it becomes a partial order (see [43]).*

Theorem 5.6. *Let $A, B \in \mathbb{C}^{n\times n}$ be of same index and suppose $\|A^{\textcircled{\dagger}}\| \leq 1$. Under these conditions, $A \leq^{\textcircled{\dagger}_p} B$ if and only if $A \leq^{\textcircled{\dagger}} B$.*

Proof. Suppose $A \leq^{\textcircled{\dagger}_p} B$ and $\|A^{\textcircled{\dagger}}\| \leq 1$, then

$$\|A^{\textcircled{\dagger}}A - A^{\textcircled{\dagger}}B\| \leq \|A^{\textcircled{\dagger}}\|\|(A - B)\| = \|A^k(A^{\textcircled{\dagger}})^{k+1}\|\|(A - B)\| \to 0 \text{ as } k \to \infty.$$

Therefore, $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$. Similarly, we can show $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.

On the other hand, if $A \leq^{\textcircled{\dagger}} B$ then

$$\begin{aligned} AA^{\textcircled{\dagger}_p} - BA^{\textcircled{\dagger}_p} &= AA^{\textcircled{\dagger}} + AA^{\textcircled{\dagger}}A - A^2A^{\textcircled{\dagger}} - BA^{\textcircled{\dagger}} - BA^{\textcircled{\dagger}}A + BAA^{\textcircled{\dagger}} \\ &= (B^2 - A^2)A^{\textcircled{\dagger}}. \end{aligned}$$

Thus,

$$\|AA^{\textcircled{\dagger}_p} - BA^{\textcircled{\dagger}_p}\| \leq \|(B^2 - A^2)A^{\textcircled{\dagger}}\| = \|(B^2 - A^2)A^k(A^{\textcircled{\dagger}})^{k+1}\| \to 0 \text{ as } k \to \infty$$

and hence $AA^{\textcircled{\dagger}_p} = BA^{\textcircled{\dagger}_p}$. Similarly, we can proof $A^{\textcircled{\dagger}_p}A = A^{\textcircled{\dagger}_p}B$. □

Remark 5.2. *The binary relation $\leq^{\textcircled{\dagger}_p}$ is a pre-order on the set $\mathcal{PO}$ defined as*

$$\mathcal{PO} = \{A \in \mathbb{C}^{n\times n} \mid \ \|A^{\textcircled{\dagger}}\| \leq 1\}.$$

6. Application in solving linear systems

In the following two results, we discuss the solution of linear certain equations using ICEP inverses, CEPI inverses and P-core-EP inverse.

Proposition 6.1. *Let $A \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$. The vector equation*

$$Az = A^k(A^k)^\dagger b \tag{6.1}$$

is consistent and the generic solution to (6.1) is

$$z = A_G^{-,\textcircled{\dagger}}b + (I_n - GA)w, \tag{6.2}$$

for any $w \in \mathbb{C}^n$.

Proof. Let $w \in \mathbb{C}^n$ and $z = A_G^{-,\textcircled{\dagger}}b + (I_n - GA)w$. Then

$$Az = A(A_G^{-,\textcircled{\dagger}}b + (I_n - GA)w) = AGAA^{\textcircled{\dagger}}b = AA^{\textcircled{\dagger}}b = AA^{\mathrm{D}}A^k(A^k)^\dagger b = A^k(A^k)^\dagger b.$$

Thus z is a solution to the equation (6.1). Suppose y is another solution to (6.1). Applying Lemma 2.2, we obtain $A_G^{-,\textcircled{\dagger}}b = GAA^{\mathrm{D}}A^k(A^k)^\dagger b = GA^k(A^k)^\dagger b = GAy$. Now

$$y = A_G^{-,\textcircled{\dagger}}b + y - A_G^{-,\textcircled{\dagger}}b = A_G^{-,\textcircled{\dagger}}b + y - GAy = A_G^{-,\textcircled{\dagger}}b + (I_n - GA)y,$$

which is of the pattern (6.2). □

Additional constraint $b \in \mathcal{R}(A^k)$ leads to the particular result given in Corollary 6.1.

Corollary 6.1. *Let* $A \triangleright_n G$ *and* $b \in \mathcal{R}(A^k)$*. Then* $Az = b$ *is unconditionally solvable and its general solution is given by*

$$z = A_G^{-,\textcircled{\dagger}} b + (I_n - GA)w = Gb + (I_n - GA)w$$

for any $w \in \mathbb{C}^n$*.*

Proof. If $b \in \mathcal{R}(A^k)$, then $b = A^k(A^k)^- b$. Such representation initiates $A_G^{-,\textcircled{\dagger}} b = GA^k(A^k)^- b = Gb$. The remaining part of the proof follows from Proposition 6.1. □

Corollary 6.2. *For* $A \triangleright_n G$*, there is the unique solution* $A_G^{-,\textcircled{\dagger}} b$ *in* $\mathcal{R}(GA^k)$ *of the system* (6.1)*.*

Proof. According to Proposition6.1 and Corollary 3.1, $A_G^{-,\textcircled{\dagger}} b$ is a solution to (6.1) in $\mathcal{R}(GA^k)$.

For uniqueness, suppose $z_1, z_2 \in \mathcal{R}(GA^k)$. Now

$$z_1 - z_2 \in \mathcal{R}(GA^k) \cap \mathcal{N}(A) \subseteq \mathcal{R}(A_G^{-,\textcircled{\dagger}} A) \cap \mathcal{N}(A_G^{-,\textcircled{\dagger}} A) = \{0\}.$$

Therefore, $z_1 = z_2$. □

Theorem 6.1. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $b \in \mathcal{R}(A)$*. Then* $A^{\textcircled{\dagger}} Az = A^{\textcircled{\dagger}} b$ *is consistent and the general solution is given by*

$$z = A_G^{\textcircled{\dagger},-} b + (I_n - A^{\textcircled{\dagger}} A)w = A^{\textcircled{\dagger}} b + (I_n - A^{\textcircled{\dagger}} A)w,$$

for any $w \in \mathbb{C}^n$*.*

Corollary 6.3. *Under the environment* $A \triangleright_n G$*, the system* $A^{\textcircled{\dagger}} Az = A^{\textcircled{\dagger}} b$ *has unique solution* $A_G^{\textcircled{\dagger},-} b$ *in* $\mathcal{R}(A^k)$*.*

6.1. Numerical examples on linear systems

The numerical examples worked out in this paper on a personal laptop with MATLAB, R2022b and the laptop with configuration: 11^{th} Gen Intel(R) Core(TM) i7-1165G7@2.80GHz, 16GB of memory, and the Microsoft Windows 11 operating system (64-bit).

The residual errors associated to different generalized inverses are summarised in below Table 2.

Table 2: Residual errors associate to generalized inverses

$E_{\dagger} = \lVert AA^{\dagger}b - b\rVert_F$	$E_D = \lVert AA^D b - b\rVert_F$	$E_{\textcircled{\dagger}} = \lVert AA^{\textcircled{\dagger}} b - b\rVert_F$
$E_{-,\textcircled{\dagger}} = \lVert AA_G^{-,\textcircled{\dagger}} b - b\rVert_F$	$E_{\textcircled{\dagger},-} = \lVert AA_G^{\textcircled{\dagger},-} b - b\rVert_F$	$E_{\textcircled{\dagger}_p} = \lVert AA^{\textcircled{\dagger}_p} b - b\rVert_F$

Example 6.1. Consider the singular matrix

$$A = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 3 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 3 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

as given in [9, p. 37]. Clearly $\operatorname{ind}(A) = 3$. By choosing 30 randomly generated vectors $b \in \mathcal{R}(A^3)$, we calculate average residual errors and the mean CPU time in seconds. It is observed that all the generalized inverses perform equally good in terms of residual error and the time complexity. The comparison details are presented in Table 3.

Table 3: Comparison analysis of residual errors and the mean CPU time

Residual error	$E_{\dagger} = 3.3277e^{-14}$	$E_D = 9.8343e^{-14}$	$E_{\textcircled{\dagger}} = 5.2683e^{-14}$
Mean CPU Time	0.002623	0.003478	0.005232
Residual error	$E_{-,\textcircled{\dagger}} = 8.3744e^{-14}$	$E_{\textcircled{\dagger},-} = 7.7856e^{-14}$	$E_{\textcircled{\dagger}_p} = 8.5431e^{-14}$
Mean CPU Time	0.005653	0.005297	0.007181

Example 6.2. Observe the elliptic partial differential Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{6.3}$$

with Neumann boundary conditions. In this equation, $f(x, y)$ denotes the input to the problem on the region $\mathbf{R} = \{(x, y) \mid 0 < x < 1,\ 0 < y < 1\}$, with the boundary $\partial\mathbf{R}$. Equations of type (6.3) came from the study of various time-invariant physical problems, such as the steady-state problems involving incomprehensible fluids. Here we use a two-dimensional adjustment of the Finite-Difference method (5-point stencil formula with uniform grid). We choose integers n to define the step sizes $h = 1/n$. The strategy to use two node points $(x_0,\ x_1)$ in X-axis and $(y_0,\ y_1)$ in Y-axis initiates generation of the coefficient matrix of the order $2^2 = 4$ and given by

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Similarly, if we consider 3 node points in each direction we will get $3^2 \times 3^2$ coefficient matrix A equal to

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}. \tag{6.4}$$

In general, n node points in each directions initiate the linear system

$$Ax = b,\ b \in \mathbb{R}^{n^2}, \tag{6.5}$$

and the matrix $A \in \mathbb{R}^{n^2 \times n^2}$ is given by

$$A = I_n \otimes P + Q \otimes I_n + D, \tag{6.6}$$

where $I_n \in \mathbb{R}^{n \times n}$ the identity matrix. Here the matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ are equal to

$$P = \begin{bmatrix} 0 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 0 \end{bmatrix} = \text{tridiagonal}\,(-1, 0, -1) = Q.$$

It is worth mentioning that $D \in \mathbb{R}^{n^2 \times n^2}$ is the diagonal matrix, and the diagonal entries change according to the number of grid points. Based on the representation (6.6) of the coefficient matrix A in (6.4), it follows ind$(A) = 1$. We choose the vector $b \in \mathcal{R}(A)$ randomly 100 times and calculate the respective residual error and the mean CPU time. The detailed comparison analysis based on a few fixed numbers of nodes in each direction are illustrated in Table 4.

Table 4: Comparison analysis of residual errors associated to generalized inverses

Order of A	Residual error	Mean CPU time (in seconds)
1600	$E_{\textcircled{\dagger}} = 4.5058e^{-11}$	3.403053
	$E_{-,\textcircled{\dagger}} = 1.6010e^{-12}$	1.946514
	$E_{\textcircled{\dagger},-} = 4.6792e^{-11}$	3.027191
	$E_{\textcircled{\dagger}_p} = 2.4573e^{-10}$	6.336402

7. Conclusion

We have introduced ICEP, the dual CEPI inverses and P-core-EP inverse on square matrices of arbitrary index. A few properties and characterizations of the these inverses have been derived. Several representations of these inverses based on core-EP decomposition and HS-decomposition is established. A binary relation is introduced for both introduced generalized inverses and main properties of this associated relation are investigated. Numerical examples carried out and an application to linear system, arises from partial differential equation illustrated. The following problems can be considered for possible future research.

- Derivation of iterative methods for computing ICEP, the dual CEPI inverses and P-core-EP inverse.
- Studying the continuity and perturbations of ICEP and CEPI inverses.
- Studying of ICEP and CEPI inverses for tensors, elements in a ring or bounded linear operators.
- Particularly, we investigate combinations of inner inverses with the core-EP inverse. Further research can include combinations of inner inverses with various kinds of outer inverses.

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