The CEPGD-inverse for square matrices

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Abstract

This paper introduces a new class of generalized inverses for square matrices: core-EP G-Drazin (CEPGD) inverse. The CEPGD inverse is not unique and defined as a proper composition of the core-EP and the G-Drazin inverse. Representations of CEPGD inverses related to the core-nilpotent decomposition and the Hartwig-Spindelböck decomposition are established. The existence of CEPGD inverses as well as a few characterizations and representations of this inverse are discussed. In addition, we consider some additional properties of the CEPGD inverses through an induced binary relation.

Keywords: Generalized inverse; Moore-Penrose inverse; Core-EP inverse; G-Drazin inverse; Matrix partial ordering.

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1. Introduction

In this section, we will restate necessary definitions, notations and known results which will be utilized to introduce and derive our main results. The notation $\mathbb{C}^{n\times n}$ denotes the complex $n\times n$ matrices. The index of $A\in\mathbb{C}^{n\times n}$, denoted by $\mathrm{ind}(A)$, is the least positive integer l which defines the border of rank-invariant powers $\mathrm{rank}(A^l)=\mathrm{rank}(A^{l+1})$. Standard notations $\mathcal{N}(A)$, A^*, and $\mathcal{R}(A)$, respectively, are used to denote the null space, conjugate transpose and the image of a complex matrix A. The orthogonal projection on $\mathcal{R}(A)$ is denoted by P_A. The Moore-Penrose inverse $A^\dagger$ of a matrix A is a distinctive solver X of the matrix system

$$(1)\ AXA=A,\quad (2)\ XAX=X,\quad (3)\ (AX)^*=AX,\quad (4)\ (XA)^*=XA.$$

In particular, if X satisfies the matrix equation $AXA=A$, then X is called an inner inverse of A. An arbitrary inner inverse of A is indicated by A^-. In addition, if $AA^-=A^-A$, we say A^- is commuting inner inverse or g-inverse of A. Similarly, if the matrix X fulfils the equation $XAX=X$, then it is denoted by $A^{(2)}$. If the range and the null space of $A^{(2)}$ are predefined as $\mathcal{R}(X)=T,\mathcal{N}(X)=S$, then such $A^{(2)}$ is denoted as $X=A^{(2)}_{T,S}$. For $A\in\mathbb{C}^{n\times n}$ satisfying $\mathrm{ind}(A)=k$, there exists the unique Drazin inverse A^{D}, satisfying

$$(1^k)\ A^{k+1}X=A^k,\quad (5)\ AX=XA,\quad (2)\ XAX=X.$$

Under the particular environment $\text{ind}(A) = 1$, the Drazin inverse becomes the group inverse $A^{\#}$.

Let $A \in \mathbb{C}^{n\times n}$ and $\text{ind}(A) = k$. Then $X \in \mathbb{C}^{n\times n}$ is called G-Drazin inverse of A if it serves as a solution for the following equations [38]

$$(1)\ AXA = A,\quad ({}^{k}1)\ XA^{k+1} = A^{k},\quad (1^{k})\ A^{k+1}X = A^{k}. \tag{1.1}$$

The G-Drazin inverse of A is denoted by A^{GD} and it is not unique and represents the set in the general case. The standard notation $A\{\text{GD}\}$ will be used to signify the set of G-Drazin inverses of A. The extension of G-Drazin inverses on rectangular matrices along with a weight matrix was discussed by Coll *et.al.* [6], and on Banach space operators in [24].

Prasad and Mohana in [19] proposed the core-EP (CEP) inverse on the set of square matrices with arbitrary index. The unique matrix X is said to be CEP inverse of A with $\text{ind}(A) = k$ if it satisfies

$$({}^{k}1)\ XA^{k+1} = A^{k},\quad (6)\ AX^{2} = X,\quad (3)\ (AX)^{*} = AX$$

and it is signified as $A^{\textcircled{\dagger}}$. The CEP inverse possesses the representation [9]

$$A^{\textcircled{\dagger}} = A^{k}(A^{k+1})^{\dagger} = A^{\text{D}}A^{k}(A^{k})^{\dagger}.$$

Further, Prasad *et al.* in [20] proposed iterations to approximate the CEP inverse. Following this research, Ferreyra *et al.* in [8] investigated some additional characterizations of the CEP inverse. Later, Zhou *et al.* in [40] discussed limit representations of the CEP inverse. Gao and Chen studied several characterizations of the CEP inverse in [10]. The CEP was extended to rectangular matrices by Ferreyra *et al.* in [8]. Several numerical methods for finding CEP inverse, theoretical studies and characterizations of the CEP inverse have been introduced recently. Certain new characterizations, representations, and perturbations of the CEP and the weighted CEP were investigated in [16, 17, 1]. A number of authors have focused on the CEP inverse and have achieved various representations. Main results are available in [9, 15, 25, 26, 32].

Now we will discuss a few composite generalized inverses which have been developed very recently. In the last few years, there has been a growing interest for developing composite generalized inverses, main of which are composite outer inverses [28, 35]. Subsequently, Hernández *et al.* in [14] introduced 2MP-inverses, MP2-inverses, and C2MP-inverses on the set of rectangular matrices.

Composite outer inverses are surveyed in Table 1.

Table 1: Survey of composite outer inverses.

Title	**Definition**	**Reference**
OMP	$A^{(2),\dagger}_{T,S} = A^{(2)}_{T,S}AA^{\dagger}$	[28]
MPO	$A^{\dagger,(2)}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}$	[28]
MPOMP	$A^{\dagger,(2),\dagger}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}AA^{\dagger}$	[28]
2MP	$A^{2MP} = A^{(2)}AA^{\dagger}$	[14]
MP2	$A^{MP2} = A^{\dagger}AA^{(2)}$	[14]
C2MP	$A^{C2MP} = A^{\dagger}AA^{(2)}AA^{\dagger}$	[14]

Composite generalized inverses have adopted in diverse areas of mathematics, including ring, matrix, Banach algebra, Hilbert space operator to extend the DMP, OMP, MPO and MPOMP inverses [4, 18, 21, 27, 28, 29, 41, 42]. Further, the DMP inverse was generalized to rectangular matrices as the W-weighted DMP inverse in [22].

A summarization of particular composite outer inverses on square matrices is presented in Table 2.

Composite one inverses were proposed in [13]. The authors of [13] introduced 1MP and MP1 generalized inverses along with studied the reduction of 1MP-inverses to partial isometries. A survey of particular composite one inverses is presented in Table 3.

Table 2: Particular cases of composite outer inverses on square matrices.

Restrictions	Title	Composite outer inverse	Reference
$\mathrm{ind}(A) = 1$	core	$A^{\text{#}} = A^{\#}AA^{\dagger}$	[1]
$\mathrm{ind}(A) = 1$	dual core	$A_{\text{#}} = A^{\dagger}AA^{\#}$	[5, 31]
-	DMP	$A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^{\dagger}$	[18, 27, 41]
-	MPD	$A^{\dagger,\mathrm{D}} = A^{\dagger}AA^{\mathrm{D}}$	[18]
-	CMP	$A^{c,\dagger} = A^{\dagger}AA^{\mathrm{D}}AA^{\dagger}$	[21, 29]
-	MPCEP	$A^{\dagger,\oplus} = A^{\dagger}AA^{\oplus}$	[4]

Table 3: Particular cases of composite one inverses.

Title	Composite one inverse	Reference
1MP	$A^{-,\dagger} = A^{-}AA^{\dagger}$	[13]
MP1	$A^{\dagger,-} = A^{\dagger}AA^{-}$	[13]
D1	$A^{\mathrm{D},-} = A^{\mathrm{D}}AA^{-}$	[33]
1D	$A^{-,\mathrm{D}} = A^{-}AA^{\mathrm{D}}$	[33]

Table 4: Particular cases of composite GD inverses.

Title	Composite one inverse	Reference
GDMP	$A^{\mathrm{GD},\dagger} = A^{\mathrm{GD}}AA^{\dagger}$	[12]
MPGD	$A^{\dagger,\mathrm{GD}} = A^{\dagger}AA^{\mathrm{GD}}$	[12]

The authors in [12] introduced the GDMP-inverse and its dual for square matrices. Definitions of such matrices are restated in in Table 4.

In addition, it is well known that generalized inverse is one of the main tools to study matrix partial order. Recently, there has been a growing interest in analyzing binary relations (reflexive, transitive, antisymmetric) on a non-empty set, such as genetics, information geometry, botanizes, data mining, physics, probability, statistics, and environmental and socioeconomic sciences (see [7, 30, 34]).

Motivated by the above mentioned various composite generalized inverses, we aim to introduce and investigate a new class of composite generalized inverses, called CEPGD inverses. This class of matrices provides a generalization of the Drazin inverses to a more general class of generalized inverses.

The main results of this paper are highlighted as in the following:

(1) A novel class of generalized inverses, termed as CEPGD inverse, is introduced.

(2) A few representations and characterizations of the CEPGD inverses are investigated.

(3) Representations of CEPGD inverses based on the core-nilpotent decomposition and the Hartwig-Spindelböck decomposition are established.

(4) Range and null space of CEPGD inverses is considered.

(5) A binary relation for these inverses is introduced along with some derived properties.

The global development of sections proceeded according to the following structure. Definition of the CEPGD inverse is given in Section 2. A few characterizations of CEPGD inverses and their relation with main existing generalized inverses and (B, C)-inverses are discussed in the same section. A binary relation on CEPGD inverses is introduced in Section 3. Last section gives some concluding remarks.

2. The CEPGD inverse

In this part, we establish the core-EP G-Drazin (CEPGD) inverse on the set of square matrices. Further, we discuss a few characterizations of CEPGD inverses and their relation with main classes of generalized inverses. From here onward, we will consider the matrix $A \in \mathbb{C}^{n\times n}$ of index $\mathrm{ind}(A) = k$.

Theorem 2.1 provides the motivation to investigate CEPGD inverses.

Theorem 2.1. *For a fixed G-Drazin inverse $A^{\mathrm{GD}} \in A\{GD\}$, the matrix expression $X = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}$ is the unique solution of the subsequent matrix equations:*

$$XAX = X, \quad XA = A^{\textcircled{\dagger}}A, \text{ and } AX = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}. \tag{2.1}$$

Proof. Let $X = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}$. Then

$$\begin{aligned} XA &= A^{\textcircled{\dagger}}AA^{\mathrm{GD}}A = A^{\textcircled{\dagger}}A, \\ AX &= AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} \end{aligned}$$

and

$$XAX = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = X.$$

Next, we will show the uniqueness of $X = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}$. Suppose there exist two solutions, say Z_1 and Z_2 satisfying equation (2.1). From the equalities $Z_1A = A^{\textcircled{\dagger}}A = Z_2A$ and $AZ_1 = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = AZ_2$, we obtain

$$Z_1 = Z_1AZ_1 = Z_2AZ_1 = Z_2AZ_2 = Z_2,$$

which completes the proof. ◻

In view of Theorem 2.1, now we define the CEPGD inverse as follows.

Definition 2.1. **(a)** *Assume that A^{GD} is an arbitrary but fixed G-Drazin inverse of A. Then the* CEPGD *inverse of A is termed as $A^{\textcircled{\dagger},\mathrm{GD}}$ and defined by the expression*

$$A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}.$$

(b) *The* CEPGD *family of A is marked with $A\{\textcircled{\dagger},\mathrm{GD}\}$ and defined as the set*

$$A\{\textcircled{\dagger},\mathrm{GD}\} = A^{\textcircled{\dagger}}AA\{\mathrm{GD}\} = \left\{A^{\textcircled{\dagger}}AA^{\mathrm{GD}} : \ A^{\mathrm{GD}} \in A\{\mathrm{GD}\}\right\}.$$

Remark 2.1. *Notice that every fixed G-Drazin inverse A^{GD} may give rise to a different* CEPGD *inverse of A. Henceforth, if we mention the* CEPGD *inverse of A, it is the* CEPGD *with previously fixed A^{GD}.*

Example 2.1. Observe the input matrix $A = \begin{bmatrix} -2 & 0 & -4 \\ 4 & 2 & 4 \\ 3 & 2 & 2 \end{bmatrix}$. Clearly, $\mathrm{rank}(A) = 2, \mathrm{rank}(A^2) = \mathrm{rank}(A^3) = 1$, so that $k = \mathrm{ind}(A) = 2$. Then

$$A^{\dagger} = \begin{bmatrix} \frac{8}{81} & \frac{7}{81} & \frac{11}{81} \\ \frac{19}{81} & \frac{13}{162} & \frac{16}{81} \\ -\frac{22}{81} & \frac{1}{81} & -\frac{10}{81} \end{bmatrix}, \quad A^{\mathrm{D}} = A^2\left(A^5\right)^{\dagger}A^2 = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$A^{\dagger,\mathrm{D}} = A^{\dagger}AA^{\mathrm{D}} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 1 & \frac{1}{2} & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^{\dagger} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 1 & \frac{1}{2} & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

$$A^{\textcircled{\dagger}} = A^2\left(A^3\right)^{\dagger} = \begin{bmatrix} \frac{2}{17} & -\frac{3}{17} & -\frac{2}{17} \\ -\frac{3}{17} & \frac{9}{34} & \frac{3}{17} \\ -\frac{2}{17} & \frac{3}{17} & \frac{2}{17} \end{bmatrix}, \quad A^{\dagger,\textcircled{\dagger}} = A^{\dagger}AA^{\textcircled{\dagger}} = \begin{bmatrix} -\frac{2}{51} & \frac{1}{17} & \frac{2}{51} \\ -\frac{1}{51} & \frac{1}{34} & \frac{1}{51} \\ -\frac{2}{51} & \frac{1}{17} & \frac{2}{51} \end{bmatrix}.$$

Further, for a fixed $A^- = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $A^{\mathrm{GD}} = \begin{bmatrix} 1 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ -2 & -3 & 3 \end{bmatrix}$, we have

$$A^{-,\mathrm{D}} = A^- A A^{\mathrm{D}} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\mathrm{D},-} = A^{\mathrm{D}} A A^- = \begin{bmatrix} -1 & -1 & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$A^{\textcircled{\dagger},-} = A^{\textcircled{\dagger}} A A^- = \begin{bmatrix} \frac{1}{17} & -\frac{5}{17} & 0 \\ -\frac{3}{34} & \frac{15}{34} & 0 \\ -\frac{1}{17} & \frac{5}{17} & 0 \end{bmatrix}, \quad A^{-,\textcircled{\dagger}} = A^- A A^{\textcircled{\dagger}} = \begin{bmatrix} -\frac{2}{17} & \frac{3}{17} & \frac{2}{17} \\ \frac{1}{17} & -\frac{3}{34} & -\frac{1}{17} \\ 0 & 0 & 0 \end{bmatrix},$$

$$A^{\dagger,\mathrm{GD}} = A^{\dagger} A A^{\mathrm{GD}} = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{3} & \frac{5}{9} \\ \frac{11}{18} & \frac{5}{6} & -\frac{5}{9} \\ -\frac{13}{9} & -\frac{7}{3} & \frac{20}{9} \end{bmatrix}, \quad A^{\mathrm{GD},\dagger} = A^{\mathrm{GD}} A A^{\dagger} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} & \frac{1}{9} \\ -\frac{1}{9} & \frac{5}{18} & \frac{2}{9} \\ -\frac{4}{9} & \frac{1}{9} & -\frac{1}{9} \end{bmatrix},$$

$$A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}} A A^{\mathrm{GD}} = \begin{bmatrix} \frac{31}{17} & \frac{55}{17} & -\frac{60}{17} \\ -\frac{93}{34} & -\frac{165}{34} & \frac{90}{17} \\ -\frac{31}{17} & -\frac{55}{17} & \frac{60}{17} \end{bmatrix}, \quad A^{\mathrm{GD},\textcircled{\dagger}} = A^{\mathrm{GD}} A A^{\textcircled{\dagger}} = \begin{bmatrix} \frac{2}{17} & -\frac{3}{17} & -\frac{2}{17} \\ -\frac{3}{17} & \frac{9}{34} & \frac{3}{17} \\ -\frac{2}{17} & \frac{3}{17} & \frac{2}{17} \end{bmatrix}.$$

It is observable that the CEPGD inverse of A differs from the selected inner-inverse, CEP inverse, Drazin inverse, Moore-Penrose inverse, G-Drazin inverse, DMP inverse, MPCEP inverse, MPD inverse, inner Drazin inverse, Drazin inner inverse, inner-core-EP, core-EP-inner inverse, GDMP-inverse, and MPGD-inverse.

Remark 2.2. *Example 2.1 revealed the identity $A^{\mathrm{GD},\textcircled{\dagger}} \equiv A^{\textcircled{\dagger}}$. The* CEPGD *inverse can be called a pre-dual core-EP inverse since the dual* GDCEP *inverse ($A^{\mathrm{GD},\textcircled{\dagger}} = A^{\mathrm{GD}} A A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$) is the same as the core-EP inverse by*

$$A^{\mathrm{GD},\textcircled{\dagger}} = A^{\mathrm{GD}} A A^{\textcircled{\dagger}} = A^{\mathrm{GD}} A^{k+1} (A^{\textcircled{\dagger}})^{k+1} = A^k (A^{\textcircled{\dagger}})^{k+1} = A (A^{\textcircled{\dagger}})^2 = A^{\textcircled{\dagger}}.$$

Example 2.2. Our goal is to continue Example 2.1 in symbolic form on the same matrix A. The solution to $AXA = A$ with X in the general form $X := \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$ gives the set of inner inverses

$$A\{1\} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & 2x_{1,1} - x_{1,2} + 2x_{2,1} - \frac{1}{2} & -2x_{1,1} - x_{1,3} - 2x_{2,1} + 1 \\ x_{3,1} & x_{1,1} - \frac{x_{1,2}}{2} + 2x_{3,1} + \frac{1}{2} & \frac{1}{2}(-2x_{1,1} - x_{1,3} - 4x_{3,1} - 1) \end{bmatrix},$$

while the general solution to (1.1) defines the set

$$A\{\mathrm{GD}\} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,1} - \frac{3x_{1,2}}{2} - \frac{1}{2} \\ \frac{1}{2} - x_{1,1} & \frac{1}{2} - x_{1,2} & \frac{1}{2}(-2x_{1,1} + 3x_{1,2} + 1) \\ x_{3,1} & x_{1,1} - \frac{x_{1,2}}{2} + 2x_{3,1} + \frac{1}{2} & \frac{1}{4}(-6x_{1,1} + 3x_{1,2} - 8x_{3,1} - 1) \end{bmatrix}.$$

Further calculation in symbolic form gives

$$A\{-,\mathrm{D}\} = A\{1\} A A^{\mathrm{D}} = \begin{bmatrix} -2x_{1,1} + 3x_{1,2} + 2x_{1,3} & -2x_{1,1} + 3x_{1,2} + 2x_{1,3} & 0 \\ 2x_{1,1} - 3x_{1,2} - 2x_{1,3} + \frac{1}{2} & 2x_{1,1} - 3x_{1,2} - 2x_{1,3} + \frac{1}{2} & 0 \\ x_{1,1} - \frac{3x_{1,2}}{2} - x_{1,3} + \frac{1}{2} & x_{1,1} - \frac{3x_{1,2}}{2} - x_{1,3} + \frac{1}{2} & 0 \end{bmatrix},$$

$$A\{\mathrm{D},-\} = A^{\mathrm{D}} A A\{1\} = \begin{bmatrix} -2(x_{1,1} + x_{2,1}) & -4x_{1,1} - 4x_{2,1} + 1 & 4x_{1,1} + 4x_{2,1} - 2 \\ 3(x_{1,1} + x_{2,1}) & 6x_{1,1} + 6x_{2,1} - \frac{3}{2} & -6x_{1,1} - 6x_{2,1} + 3 \\ 2(x_{1,1} + x_{2,1}) & 4x_{1,1} + 4x_{2,1} - 1 & -4x_{1,1} - 4x_{2,1} + 2 \end{bmatrix},$$

$$A\{\textcircled{\dagger},-\} = A^{\textcircled{\dagger}}AA\{1\} =$$

$$\begin{bmatrix} -\frac{2}{17}\left(11x_{1,1}+5x_{2,1}+12x_{3,1}\right) & \frac{1}{17}\left(-44x_{1,1}-20x_{2,1}-48x_{3,1}-7\right) & \frac{2}{17}\left(22x_{1,1}+10x_{2,1}+24x_{3,1}+1\right) \\ \frac{3}{17}\left(11x_{1,1}+5x_{2,1}+12x_{3,1}\right) & \frac{3}{34}\left(44x_{1,1}+20x_{2,1}+48x_{3,1}+7\right) & -\frac{3}{17}\left(22x_{1,1}+10x_{2,1}+24x_{3,1}+1\right) \\ \frac{2}{17}\left(11x_{1,1}+5x_{2,1}+12x_{3,1}\right) & \frac{1}{17}\left(44x_{1,1}+20x_{2,1}+48x_{3,1}+7\right) & -\frac{2}{17}\left(22x_{1,1}+10x_{2,1}+24x_{3,1}+1\right) \end{bmatrix},$$

$$A\{-,\textcircled{\dagger}\} = A\{1\}AA^{\textcircled{\dagger}}$$

$$= \begin{bmatrix} \frac{2}{17}\left(2x_{1,1}-3x_{1,2}-2x_{1,3}\right) & \frac{3}{17}\left(-2x_{1,1}+3x_{1,2}+2x_{1,3}\right) & \frac{2}{17}\left(-2x_{1,1}+3x_{1,2}+2x_{1,3}\right) \\ \frac{1}{17}\left(-4x_{1,1}+6x_{1,2}+4x_{1,3}-1\right) & \frac{3}{34}\left(4x_{1,1}-6x_{1,2}-4x_{1,3}+1\right) & \frac{1}{17}\left(4x_{1,1}-6x_{1,2}-4x_{1,3}+1\right) \\ \frac{1}{17}\left(-2x_{1,1}+3x_{1,2}+2x_{1,3}-1\right) & \frac{3}{34}\left(2x_{1,1}-3x_{1,2}-2x_{1,3}+1\right) & \frac{1}{17}\left(2x_{1,1}-3x_{1,2}-2x_{1,3}+1\right) \end{bmatrix},$$

$$A^{\dagger,\mathrm{GD}} = A^{\dagger}AA\{\mathrm{GD}\} = \begin{bmatrix} \frac{1}{9}\left(x_{1,1}+2x_{3,1}+2\right) & \frac{1}{9}\left(2x_{1,1}+4x_{3,1}+3\right) & \frac{1}{9}\left(-2x_{1,1}-4x_{3,1}-1\right) \\ \frac{1}{18}\left(-2x_{1,1}-4x_{3,1}+5\right) & \frac{1}{18}\left(-4x_{1,1}-8x_{3,1}+3\right) & \frac{1}{9}\left(2x_{1,1}+4x_{3,1}+1\right) \\ \frac{1}{9}\left(4x_{1,1}+8x_{3,1}-1\right) & \frac{1}{9}\left(8x_{1,1}+16x_{3,1}+3\right) & -\frac{4}{9}\left(2x_{1,1}+4x_{3,1}+1\right) \end{bmatrix},$$

$$A^{\mathrm{GD},\dagger} = A\{\mathrm{GD}\}AA^{\dagger} = \begin{bmatrix} \frac{1}{9}\left(10x_{1,1}-5x_{1,2}-1\right) & \frac{1}{9}\left(2x_{1,1}-x_{1,2}-2\right) & \frac{1}{18}\left(14x_{1,1}-7x_{1,2}-5\right) \\ \frac{1}{9}\left(-10x_{1,1}+5x_{1,2}+4\right) & \frac{1}{18}\left(-4x_{1,1}+2x_{1,2}+7\right) & \frac{1}{18}\left(-14x_{1,1}+7x_{1,2}+11\right) \\ \frac{1}{18}\left(-10x_{1,1}+5x_{1,2}-3\right) & \frac{1}{18}\left(-2x_{1,1}+x_{1,2}+3\right) & \frac{1}{36}\left(-14x_{1,1}+7x_{1,2}+3\right) \end{bmatrix},$$

$$A\{\textcircled{\dagger},\mathrm{GD}\} = A^{\textcircled{\dagger}}AA\{\mathrm{GD}\}$$

$$= \begin{bmatrix} \frac{1}{17}\left(-12x_{1,1}-24x_{3,1}-5\right) & \frac{1}{17}\left(-24x_{1,1}-48x_{3,1}-17\right) & \frac{12}{17}\left(2x_{1,1}+4x_{3,1}+1\right) \\ \frac{3}{34}\left(12x_{1,1}+24x_{3,1}+5\right) & \frac{3}{34}\left(24x_{1,1}+48x_{3,1}+17\right) & -\frac{18}{17}\left(2x_{1,1}+4x_{3,1}+1\right) \\ \frac{1}{17}\left(12x_{1,1}+24x_{3,1}+5\right) & \frac{1}{17}\left(24x_{1,1}+48x_{3,1}+17\right) & -\frac{12}{17}\left(2x_{1,1}+4x_{3,1}+1\right) \end{bmatrix},$$

$$A\{\mathrm{GD},\textcircled{\dagger}\} = A\{\mathrm{GD}\}AA^{\textcircled{\dagger}}$$

$$= \begin{bmatrix} \frac{2}{17} & -\frac{3}{17} & -\frac{2}{17} \\ -\frac{3}{17} & \frac{9}{34} & \frac{3}{17} \\ -\frac{2}{17} & \frac{3}{17} & \frac{2}{17} \end{bmatrix}.$$

The conclusion is that the CEPGD inverse of A is different from main classes of generalized inverses. In addition, $A\{\mathrm{GD},\textcircled{\dagger}\} = \{A^{\textcircled{\dagger}}\}$.

An equivalent definition of the CEPGD inverse is discussed in the following theorem.

Theorem 2.2. *For arbitrary* $A^{\mathrm{GD}} \in A\{\mathrm{GD}\}$*, the* CEPGD *inverse* $A^{\textcircled{\dagger},\mathrm{GD}}$ *is the unique solution to the following constrained matrix equations:*

(i) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*AA^{\mathrm{GD}})}$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$*, where* $P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*AA^{\mathrm{GD}})}$ *is a projection onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}((A^k)^*AA^{\mathrm{GD}})$*;*

(ii) $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ *and* $\mathcal{R}(X^*) \subseteq \mathcal{R}((AA^{\mathrm{GD}})^*)$*, where* $P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ *is a projection onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}((A^k)^*A)$*.*

Proof. (i) The first condition $AA^{\textcircled{\dagger},\mathrm{GD}} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*AA^{\mathrm{GD}})}$ follows from $A^{\textcircled{\dagger}} = A^{\mathrm{D}}A^k(A^k)^{\dagger}$ and

$$AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^k(A^k)^{\dagger}AA^{\mathrm{GD}}.$$

From $\mathcal{R}(A^k) = \mathcal{R}(A^{\textcircled{\dagger}}) \supseteq \mathcal{R}(A^{\textcircled{\dagger}}AA^{\mathrm{GD}}) = \mathcal{R}(A^{\textcircled{\dagger},\mathrm{GD}})$, it follows that $A^{\textcircled{\dagger},\mathrm{GD}}$ is the solution to the equation (i).

It remains to verify that the equation (i) is uniquely solvable. Suppose the existence of different solutions X_1 and X_2 to the equation (i). Then

$$A(X_1 - X_2) = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^* AA^{\mathrm{GD}})} - P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^* AA^{\mathrm{GD}})} = 0.$$

Consequently $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A) = \mathcal{N}(A^{\textcircled{\dagger}} A)$. Further, using

$$\mathcal{R}(X_1) \subseteq \mathcal{R}(A^k) = \mathcal{R}(A^{\textcircled{\dagger}} A) \text{ and } \mathcal{R}(X_2) \subseteq \mathcal{R}(A^{\textcircled{\dagger}} A),$$

we conclude $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^{\textcircled{\dagger}} A) \cap \mathcal{N}(A^{\textcircled{\dagger}} A) = \{0\}$. Consequently, $X_1 = X_2$ and hence $A^{\textcircled{\dagger},\mathrm{GD}}$ is the unique solution to the equation (i).

(ii) Notice that

$$A^{\textcircled{\dagger},\mathrm{GD}} A = A^{\textcircled{\dagger}} A = A^D A^k (A^k)^{\dagger} A = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^* A)}$$

and

$$\mathcal{R}((A^{\textcircled{\dagger},\mathrm{GD}})^*) = \mathcal{R}((A^{\textcircled{\dagger}} AA^{\mathrm{GD}})^*) \subseteq \mathcal{R}((AA^{\mathrm{GD}})^*).$$

Hence, (ii) has the solution $A^{\textcircled{\dagger},\mathrm{GD}}$. If X_1 and X_2 are two solution of (ii), we observe

$$\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{N}(A^*) \cap \mathcal{R}((AA^{\mathrm{GD}})^*) \subseteq \mathcal{N}((A^{\mathrm{GD}})^* A^*) \cap \mathcal{R}((A^{\mathrm{GD}})^* A^*) = \{0\},$$

which implies $X_1 = X_2$. □

2.1. Characterization of CEPGD Inverses

In the first result, we discuss a few properties of the CEPGD inverses, which can be verified easily.

Proposition 2.1. *For each $A^{\mathrm{GD}} \in \mathcal{A}\{GD\}$, the* CEPGD *inverse $A^{\textcircled{\dagger},\mathrm{GD}}$ satisfies the subsequent properties:*

(i) $A^{\textcircled{\dagger},\mathrm{GD}} A = A^{\textcircled{\dagger}} A$.

(ii) $A^{\textcircled{\dagger},\mathrm{GD}} A^{k+1} = A^k$, $k = \mathrm{ind}(A)$.

(iii) $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger},GD} AA^{\mathrm{GD}}$.

(iv) $A^{\textcircled{\dagger},\mathrm{GD}} AA^{\textcircled{\dagger},GD} = A^{\textcircled{\dagger},GD}$.

(v) $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\mathrm{D}} A^l (A^l)^{\dagger} AA^{\mathrm{GD}} = A^{\mathrm{D}} P_{\mathcal{R}(A^l)} AA^{\mathrm{GD}}$, *where* $l \geq k = \mathrm{ind}(A)$.

Theorem 2.3 characterizes the GDCEP inverse from an alternative algebraic access.

Theorem 2.3. *For $A^{\mathrm{GD}} \in A\{GD\}$, the subsequent matrix systems are equivalent:*

(i) $X = A^{\textcircled{\dagger},\mathrm{GD}}$.

(ii) $XAX = X$, $AX = AA^{\textcircled{\dagger}} AA^{\mathrm{GD}}$, $XA = A^{\textcircled{\dagger}} A$ *and* $AXA = AA^{\textcircled{\dagger}} A$.

(iii) $XA = A^{\textcircled{\dagger}} A$ *and* $XAA^{\textcircled{\dagger}} AA^{\mathrm{GD}} = X$.

(iv) $AX = AA^{\textcircled{\dagger}} AA^{\mathrm{GD}}$ *and* $A^{\textcircled{\dagger}} AX = X$.

(v) $XAA^{\textcircled{\dagger}} AX = X$, $XAA^{\textcircled{\dagger}} A = A^{\textcircled{\dagger}} A$, $AA^{\textcircled{\dagger}} AX = AA^{\textcircled{\dagger}} AA^{\mathrm{GD}}$ *and* $AA^{\textcircled{\dagger}} AXAA^{\textcircled{\dagger}} A = AA^{\textcircled{\dagger}} A$.

(vi) $XAA^{\textcircled{\dagger}} AX = X$, $XAA^{\textcircled{\dagger}} A = A^{\textcircled{\dagger}} A$ *and* $AA^{\textcircled{\dagger}} AX = AA^{\textcircled{\dagger}} AA^{\mathrm{GD}}$.

(vii) $XAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$ *and* $XAA^{\textcircled{\dagger}} AA^{\mathrm{GD}} = X$.

(viii) $XAA^{\dagger} = A^{\textcircled{\dagger}}$ *and* $XAA^{\textcircled{\dagger}} AA^{\mathrm{GD}} = X$.

(ix) $XAA^* = A^{\textcircled{\dagger}}AA^*$ *and* $XAA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = X$.

(x) $A^\dagger AX = A^{\dagger,\textcircled{\dagger}}AA^{\mathrm{GD}}$ *and* $A^{\textcircled{\dagger}}AX = X$.

(xi) $A^*AX = A^*AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$ *and* $A^{\textcircled{\dagger}}AX = X$.

(xii) $AX = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$ *and* $A^{\textcircled{\dagger}}AXAA^{\mathrm{GD}} = X$.

(xiii) $XA = A^{\textcircled{\dagger}}A$ *and* $A^{\textcircled{\dagger}}AXAA^{\mathrm{GD}} = X$.

(xiv) $AXA = AA^{\textcircled{\dagger}}A$ *and* $A^{\textcircled{\dagger}}AXAA^{\mathrm{GD}} = X$.

Proof. (i)⇒(ii) It is sufficient to show only $AXA = AA^{\textcircled{\dagger}}A$. Using $A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = X$, we get $AXA = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}A = AA^{\textcircled{\dagger}}A$.
(ii)⇒(iii) From $AX = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$, we obtain $XAA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = XAX = X$.
(iii)⇒(i) This implication is confirmed by $X = XAA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}$.
(ii)⇒(iv) Since $XA = A^{\textcircled{\dagger}}A$, then $A^{\textcircled{\dagger}}AX = XAX = X$.
(iv)⇒(i) Using $AX = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$, it is concluded

$$X = A^{\textcircled{\dagger}}AX = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}.$$

(i)⇒(v) From $A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = X$, we obtain

$$\begin{aligned} XAA^{\textcircled{\dagger}}A &= A^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A, \\ AA^{\textcircled{\dagger}}AX &= AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}, \\ AA^{\textcircled{\dagger}}A(XAA^{\textcircled{\dagger}}A) &= AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A, \end{aligned}$$

and

$$(XAA^{\textcircled{\dagger}}A)X = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = X.$$

(v)⇒(vi) The proof is obvious.
(vi)⇒(i) This statement follows from the below expression:

$$A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AX = A^{\textcircled{\dagger}}AX = XAA^{\textcircled{\dagger}}AX = X.$$

The remainder of the proof is completed similarly. □

Proposition 2.2. *Assume* $X \in \mathbb{C}^{n\times n}$ *and* $A^{\mathrm{GD}} \in A\{GD\}$. *Then*

(i) $AA^{\textcircled{\dagger}}AX = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} \Longleftrightarrow A^{\textcircled{\dagger}}AX = A^{\textcircled{\dagger},\mathrm{GD}}$.

(ii) $XAA^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A \Longleftrightarrow XAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} \Longleftrightarrow XA^k = A^{\textcircled{\dagger}}A^k$.

(iii) $AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\mathrm{GD}} \Longleftrightarrow A = AA^{\textcircled{\dagger}}A \Longleftrightarrow A^\dagger = A^{\dagger,\textcircled{\dagger}} \Longleftrightarrow A^* = A^*AA^{\textcircled{\dagger}}$.

Proof. (i) Let $AA^{\textcircled{\dagger}}AX = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$. Then

$$A^{\textcircled{\dagger}}AX = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AX = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger},\mathrm{GD}}.$$

The converse part is trivial.
(ii) The first part follows from $XAA^{\textcircled{\dagger}} = XAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$. To show the next equivalent statement, let $XAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$. Then

$$XA^k = XA^{\textcircled{\dagger}}A^{k+1} = XA(A^{\textcircled{\dagger}})^2A^{k+1} = XAA^{\textcircled{\dagger}}A^k = A^{\textcircled{\dagger}}A^k.$$

Conversely, if $XA^k = A^{\textcircled{\dagger}}A^k$, it follows

$$A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^k(A^{\textcircled{\dagger}})^k = XA^k(A^{\textcircled{\dagger}})^k = XAA^{\textcircled{\dagger}}.$$

(iii) Under the assumption $AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\mathrm{GD}}$, one obtains

$$A = AA^{\mathrm{GD}}A = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}A = AA^{\textcircled{\dagger}}A.$$

Vice versa, let $AA^{\textcircled{\dagger}}A = A$. In this case, $AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = AA^{\mathrm{GD}}$. □

Next, we discuss the relations of the CEPGD inverse with CEP, G-Drazin, and (B,C)-inverse.

Proposition 2.3. *Let $A^{\mathrm{GD}} \in A\{GD\}$. The* CEPGD *inverse satisfies the subsequent relations:*

(i) $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\mathrm{GD}} \Longleftrightarrow \mathcal{R}(A^{\mathrm{GD}}) \subseteq \mathcal{R}(A^{\textcircled{\dagger}})$.

(ii) $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}} \Longleftrightarrow AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\textcircled{\dagger}}$.

Proof. (i) Assume $R(A^{\mathrm{GD}}) \subseteq R(A^{\textcircled{\dagger}})$. Then $A^{\mathrm{GD}} = A^{\textcircled{\dagger}}Z$ for a selected $Z \in \mathbb{C}^{n\times n}$. Now

$$A^{\mathrm{GD}} = A^{\textcircled{\dagger}}Z = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}Z = A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^{\textcircled{\dagger},\mathrm{GD}}.$$

The converse is trivial.
(ii) Let $AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\textcircled{\dagger}}$. Then $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$. The converse is trivial. □

Recall that A is called index EP (in short, i-EP) if $A^k(A^k)^\dagger = (A^k)^\dagger A^k$.

Theorem 2.4. *If A is i-EP, then $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}}$. Moreover, $A^{\textcircled{\dagger},\mathrm{GD}} = A^{\textcircled{\dagger}} = A^{\mathrm{D}}$.*

Proof. Let A be i-EP. Then by [39, Theorem 2.4], it follows $A^{\textcircled{\dagger}} = A^D$ and $AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A$. Now,

$$\begin{aligned} A^{\textcircled{\dagger},\mathrm{GD}} &= A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = AA^{\textcircled{\dagger}}A^{\mathrm{GD}} \\ &= A^{k+1}(A^{\textcircled{\dagger}})^{k+1}A^{\mathrm{GD}} = (A^{\textcircled{\dagger}})^{k+1}A^{k+1}A^{\mathrm{GD}} \\ &= (A^{\textcircled{\dagger}})^{k+1}A^k = A^k(A^{\textcircled{\dagger}})^{k+1} = A(A^{\textcircled{\dagger}})^2 \\ &= A^{\textcircled{\dagger}}, \end{aligned}$$

which was our initial intention. □

Definition 2.2 restates definition of (B,C)-inverses.

Definition 2.2. [2, 3] *Let $A, B, C \in \mathbb{C}^{n\times n}$. A unique matrix $X \in \mathbb{C}^{n\times n}$ is called the (B,C)-inverse of A if it satisfies*

$$XAB = B, \quad CAX = C, \quad \mathcal{N}(C) = \mathcal{N}(X), \quad \mathcal{R}(X) = \mathcal{R}(B).$$

Theorem 2.5 and Corollary 2.1 provide useful representations of the CEPGD inverse in terms of (B,C)-inverses.

Theorem 2.5. *For arbitrary $A^{\mathrm{GD}} \in A\{GD\}$, $A^{\textcircled{\dagger},\mathrm{GD}}$ represents the (B,C) inverse of A, where $B = A^k$ and $C = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$.*

Proof. Let $X = A^{\textcircled{\dagger},\mathrm{GD}}$, $B = A^k$ and $C = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$. Then

$$\begin{aligned} XAB &= A^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^k = A^{\textcircled{\dagger}}A^{k+1} = A^k = B \\ CAX &= AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = C. \end{aligned}$$

From $X = A^{\textcircled{\dagger}}AA^{\mathrm{GD}} = A^k(A^{\textcircled{\dagger}})^{k+1}AA^{\mathrm{GD}}$, it follows $\mathcal{R}(X) \subseteq \mathcal{R}(B)$. Similarly, using $B = A^k = A^{\textcircled{\dagger}}A^{k+1} = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}A^{k+1} = XAA^{k+1}$, we conclude $\mathcal{R}(B) \subseteq \mathcal{R}(X)$. Next we need to show $\mathcal{N}(X) = \mathcal{N}(C)$. Let $y \in \mathcal{N}(C)$. Then $y \in \mathcal{N}(AA^{\textcircled{\dagger}}AA^{\mathrm{GD}})$ and subsequently, $AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}y = 0$. Premultiplying $A^{\textcircled{\dagger}}$ on both sides we get $A^{\textcircled{\dagger}}AA^{\mathrm{GD}}y = 0$. Now $Xy = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}y = 0$. Thus $y \in \mathcal{N}(X)$ and hence $\mathcal{N}(C) \subseteq \mathcal{N}(X)$. Conversely, if $z \in \mathcal{N}(X)$, then $A^{\textcircled{\dagger}}AA^{\mathrm{GD}}z = 0$. Now $Cz = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}z = 0$ and consequently, $z \in \mathcal{N}(C)$. Hence $\mathcal{N}(X) \subseteq \mathcal{N}(C)$, which completes the proof. □

Corollary 2.1. *For an arbitrary $A^{\mathrm{GD}} \in A\{GD\}$, $A^{\textcircled{\dagger},\mathrm{GD}}$ is the (B,C) inverse of A, such that $B = A^{\textcircled{\dagger}}$ and $C = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}$.*

Proof. Let $X := A^{\textcircled{\dagger},\mathrm{GD}}$. The proof of the identity $XAB = B$ follows from the following identities:

$$XAB = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} = B,$$

The identity $CAX = C$ follows from Theorem 2.5. □

Corollary 2.2. *For arbitrary $A^{\mathrm{GD}} \in A\{\mathrm{GD}\}$, the subsequent representations are valid:*

$$\begin{aligned} A^{\textcircled{\dagger},\mathrm{GD}} &= A^k\left(AA^{\textcircled{\dagger}}A^{k+1}\right)^{\dagger}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} \\ &= A^k\left(A^{k+1}\right)^{\dagger}AA^{\mathrm{GD}}; \end{aligned}$$

Proof. Stated representations follow from Theorem 2.5, and the general representation of outer inverses with known image and kernel from [36]. Based on Theorem 2.5, we obtain

$$\begin{aligned} A^{\textcircled{\dagger},\mathrm{GD}} &= A^k\left(AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^k\right)^{\dagger}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} \\ &= A^k\left(AA^{\textcircled{\dagger}}A^{k+1}\right)^{\dagger}AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} \\ &= A^k\left(AA^{\textcircled{\dagger}}A^{k+1}\right)^{\dagger}AA^{\textcircled{\dagger},\mathrm{GD}}. \end{aligned}$$

The second representation follows from $A^{\textcircled{\dagger}} = A^k(A^{k+1})^{\dagger}$, which implies

$$\begin{aligned} A^{\textcircled{\dagger},\mathrm{GD}} &= A^k\left(AA^k(A^{k+1})^{\dagger}A^{k+1}\right)^{\dagger}AA^k(A^{k+1})^{\dagger}AA^{\mathrm{GD}} \\ &= A^k\left(A^{k+1}\right)^{\dagger}A^{k+1}(A^{k+1})^{\dagger}AA^{\mathrm{GD}} \\ &= A^k\left(A^{k+1}\right)^{\dagger}AA^{\mathrm{GD}}, \end{aligned}$$

and the proof is complete. □

Example 2.3. *Consider the input matrix*

$$A = \begin{bmatrix} -2a & 0 & -4a \\ 4a & 2a & 4a \\ 3a & 2a & 2a \end{bmatrix},$$

where a is an unevaluated variable with real values. The general solution to (1.1) gives the set

$$A\{\mathrm{GD}\} = \begin{bmatrix} \frac{3x_{1,2}}{2} + x_{1,3} + \frac{1}{2a} & x_{1,2} & x_{1,3} \\ -\frac{3x_{1,2}}{2} - x_{1,3} & \frac{1}{2a} - x_{1,2} & -x_{1,3} \\ -\frac{ax_{1,2}+ax_{1,3}-ax_{3,2}+1}{2a} & x_{3,2} & \frac{1}{2}\left(-x_{1,2}-x_{1,3}-2x_{3,2}\right) \end{bmatrix},$$

which generates the set of CEPGD inverses of A:

$$\begin{aligned} A\{\textcircled{\dagger},\mathrm{GD}\} &= A^{\textcircled{\dagger}}AA\{\mathrm{GD}\} \\ &= \begin{bmatrix} \frac{-6ax_{1,2}-12ax_{3,2}+1}{17a} & -\frac{12ax_{1,2}+24ax_{3,2}+5}{17a} & \frac{12}{17}\left(x_{1,2}+2x_{3,2}\right) \\ \frac{3(6ax_{1,2}+12ax_{3,2}-1)}{34a} & \frac{3(12ax_{1,2}+24ax_{3,2}+5)}{34a} & -\frac{18}{17}\left(x_{1,2}+2x_{3,2}\right) \\ \frac{6ax_{1,2}+12ax_{3,2}-1}{17a} & \frac{12ax_{1,2}+24ax_{3,2}+5}{17a} & -\frac{12}{17}\left(x_{1,2}+2x_{3,2}\right) \end{bmatrix}. \end{aligned}$$

The matrices B and C from Theorem 2.5 are equal to

$$B := A^2 = \begin{bmatrix} -8a^2 & -8a^2 & 0 \\ 12a^2 & 12a^2 & 0 \\ 8a^2 & 8a^2 & 0 \end{bmatrix}$$

and

$$\begin{aligned} C &:= AA^{\oplus}AA\{\mathrm{GD}\} \\ &= \begin{bmatrix} -\frac{2}{17}(6ax_{1,2}+12ax_{3,2}-1) & -\frac{2}{17}(12ax_{1,2}+24ax_{3,2}+5) & \frac{24}{17}a(x_{1,2}+2x_{3,2}) \\ \frac{3}{17}(6ax_{1,2}+12ax_{3,2}-1) & \frac{3}{17}(12ax_{1,2}+24ax_{3,2}+5) & -\frac{36}{17}a(x_{1,2}+2x_{3,2}) \\ \frac{2}{17}(6ax_{1,2}+12ax_{3,2}-1) & \frac{2}{17}(12ax_{1,2}+24ax_{3,2}+5) & -\frac{24}{17}a(x_{1,2}+2x_{3,2}) \end{bmatrix}. \end{aligned}$$

Symbolic calculation confirms that $B(CAB)^{\dagger}C$, which is in accordance with Theorem 2.2.

The core-EP of A in symbolic form is given by the matrix

$$A^{\oplus} = A^2\left(A^3\right)^{\dagger} = \begin{bmatrix} \frac{2}{17a} & -\frac{3}{17a} & -\frac{2}{17a} \\ -\frac{3}{17a} & \frac{9}{34a} & \frac{3}{17a} \\ -\frac{2}{17a} & \frac{3}{17a} & \frac{2}{17a} \end{bmatrix}.$$

Using $B_1 := A^{\oplus}$ and $C := AA^{\oplus}AA\{\mathrm{GD}\}$ we conclude $B(CAB)^{\dagger}C = B_1(CAB_1)^{\dagger}C$, which is a confirmation of Corollary 2.2.

Theorem 2.6. *For $M \in \mathbb{C}^{n\times n}$ and $A^{\mathrm{GD}} \in A\{\mathrm{GD}\}$, the next statements are mutually equivalent:*

(i) $A^{\oplus,\mathrm{GD}} = MAA^{\mathrm{GD}}$.

(ii) $MA = A^{\oplus}A$.

(iii) $AMA = AA^{\oplus}A$ *and* $\mathcal{R}(MA) = \mathcal{R}(A^k)$.

(iv) $M = A^{\oplus} + Z(I_n - AA^{\mathrm{GD}})$ *for some* $Z \in \mathbb{C}^{n\times n}$.

Proof. (i)⇒(ii) Let $A^{\oplus,\mathrm{GD}} = MAA^{\mathrm{GD}}$. Then $MA = MAA^{\mathrm{GD}}A = A^{\oplus,\mathrm{GD}}A = A^{\oplus}A$.
(ii)⇒(iii) From, $MA = A^{\oplus}A$, and $A^{\oplus} = A^{\oplus}AA^{\oplus} = MAA^{\oplus}$, we obtain $\mathcal{R}(MA) = \mathcal{R}(A^{\oplus}) = \mathcal{R}(A^k)$.
(iii)⇒(i) Let $\mathcal{R}(MA) = \mathcal{R}(A^k) = \mathcal{R}(A^{\oplus})$. Then $MA = A^{\oplus}X$ for appropriate $X \in \mathbb{C}^{n\times n}$, which initiates

$$MAA^{\mathrm{GD}} \;=\; A^{\oplus}XA^{\mathrm{GD}} = A^{\oplus}AMAA^{\mathrm{GD}} = A^{\oplus}AA^{\oplus}AA^{\mathrm{GD}} = A^{\oplus,\mathrm{GD}}.$$

(ii)⇒(iv) We can easily verify that the general solution to the homogeneous equation $MA = 0$ is equal to $M = Z(I_n - AA^{\mathrm{GD}})$, where $Z \in \mathbb{C}^{n\times n}$ be arbitrary. Since $A^{\oplus}$ is a particular solution of $MA = A^{\oplus}A$, so the general solution to $MA = A^{\oplus}A$ is equal to

$$M = A^{\oplus} + Z(I_n - AA^{\mathrm{GD}}), \text{ where } Z \in \mathbb{C}^{n\times n} \text{ be arbitrary.}$$

(iv)⇒(i) Let $M = A^{\oplus} + Z(I_n - AA^{\mathrm{GD}})$ for a chosen $Z \in \mathbb{C}^{n\times n}$. In this case,

$$MAA^{\mathrm{GD}} = A^{\oplus}AA^{\mathrm{GD}} + ZAA^{\mathrm{GD}} - ZAA^{\mathrm{GD}}AA^{\mathrm{GD}} = A^{\oplus}AA^{\mathrm{GD}} = A^{\oplus,\mathrm{GD}}.$$

The proof is completed. □

Theorem 2.7. *For arbitrary $M \in \mathbb{C}^{n\times n}$ and $A^{\mathrm{GD}} \in A\{GD\}$, the subsequent assertions are valid:*

(i) $A^{\oplus,\mathrm{GD}} = A^{\oplus}AM \iff M = A^{\mathrm{GD}} + (I_n - A^{\oplus}A)Y$ *for a hosen* $Y \in \mathbb{C}^{n\times n}$.

(ii) $A^{\oplus,\mathrm{GD}} = ZAM \iff M = A^{\mathrm{GD}} + (I_n - A^{\oplus}A)Y$ *and* $Z = A^{\oplus} + X(I_n - AA^{\mathrm{GD}})$ *for selected* $X, Y \in \mathbb{C}^{n\times n}$.

Proof. (i) For $M = A^{\mathrm{GD}} + (I_n - A^{\oplus}A)Y$ it can be verified $A^{\oplus}AM = A^{\oplus,\mathrm{GD}}$. Conversely, let $A^{\oplus,\mathrm{GD}} = A^{\oplus}AM$. It is known that A^{GD} is a particular solution of $A^{\oplus}AM = A^{\oplus,\mathrm{GD}}$. Suppose Y is any solution of the homogeneous equation $A^{\oplus}AM = 0$. Then $A^{\oplus}AY = 0$, and we can write $Y = Y - A^{\oplus}AY = (I_n - A^{\oplus}A)Y$. So the general solution of homogeneous equation $A^{\oplus}AM = 0$ is given by $M = (I_n - A^{\oplus}A)Y$ and consequently, $M = A^{\mathrm{GD}} + (I_n - A^{\oplus}A)Y$ is the general solution of $A^{\oplus}AM = A^{\oplus,\mathrm{GD}}$, where $Y \in \mathbb{C}^{n\times n}$ is arbitrary.
(ii) This follows directly from part (*i*) and (*iv*) of Theorem 2.6. □

Next, we provide a representation of the CEP inverse and the GD inverse based on the core-EP matrix decomposition. More precisely, the core-EP decomposition [37] of $A \in \mathbb{C}^{n\times n}$ is defined by

$$A = U \begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^*, \tag{2.2}$$

where U is unitary, $T_1 \in \mathbb{C}^{r\times r}$ is a non-singular and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is a nilpotent matrix. Using the decomposition given in (2.2), the following results are obtained.

Theorem 2.8. *Observe the core-EP decomposition of A defined in* (2.2). *Then all* CEPGD *inverses of A are represented by*

$$A^{\textcircled{\dagger},\mathrm{GD}} = U \begin{bmatrix} T_1^{-1} & X_2 + T_1^{-1}T_2N^- \\ 0 & 0 \end{bmatrix} U^*,$$

where $(T_1X_2 + T_2N^-)N = 0$, $X_2 = T_1^{-(k+1)}\tilde{T}_2 - T_1^{-k}\tilde{T}_2N^-$ and $\tilde{T}_2 = \sum\limits_{i=0}^{k-1} T_1^i\ T_2\ N^{k-1-i}$.

Proof. Let $A = U \begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^*$. From $A = AXA$, we obtain

$$\begin{aligned} U \begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^* &= U \begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ 0 & N \end{bmatrix} U^* \\ &= U \begin{bmatrix} T_1X_1T_1 + T_2X_3T_1 & T_1X_1T_2 + T_1X_2N + T_2X_3T_2 + T_2X_4N \\ NX_3T_1 & NX_3T_2 + NX_4N \end{bmatrix} U^*. \end{aligned}$$

Thus, $NX_3 = 0,\ X_4 = N^-,\ T_1X_1 + T_2X_3 = I,\ (T_1X_2 + T_2X_4)N = 0$. Next, we evaluate

$$A^k = U \begin{bmatrix} T_1^k & \tilde{T}_2 \\ 0 & 0 \end{bmatrix} U^* \text{ (with } \tilde{T}_2 = \sum_{i=0}^{k-1} T_1^i\ T_2\ N^{k-1-i}),$$

$$XA^k = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} T_1^k & \tilde{T}_2 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} X_1T_1^k & X_1\tilde{T}_2 \\ X_3T_1^k & X_3\tilde{T}_2 \end{bmatrix} U^*,$$

and

$$A^kX = U \begin{bmatrix} T_1^k & \tilde{T}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^* = U \begin{bmatrix} T_1^kX_1 + \tilde{T}_2X_3 & T_1^kX_2 + \tilde{T}_2X_4 \\ 0 & 0 \end{bmatrix} U^*.$$

Using the condition $XA^k = A^kX$, we obtain $X_1T_1^k = T_1^kX_1,\ X_3 = 0,\ X_1\tilde{T}_2 = T_1^kX_2 + \tilde{T}_2X_4$. Hence, GD inverses of A are of the form

$$A^{\mathrm{GD}} = U \begin{bmatrix} T_1^{-1} & X_2 \\ 0 & N^- \end{bmatrix} U^*, \tag{2.3}$$

where $(T_1X_2 + T_2N^-)N = 0$ and $X_2 = T_1^{-(k+1)}\tilde{T}_2 - T_1^{-k}\tilde{T}_2N^-$. In [37], the core-EP decomposition of A is given by

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2.4}$$

From the equations (2.2), (2.3) and (2.4), we can see that CEPGD inverses are of the form

$$A^{\textcircled{\dagger},\mathrm{GD}} = U \begin{bmatrix} T_1^{-1} & X_2 + T_1^{-1}T_2N^- \\ 0 & 0 \end{bmatrix} U^*.$$

□

2.2. CEPGD inverse in terms of HS-decomposition

We discuss the general form of CEPGD inverses via the Hartwig and Spindelböck decomposition (in short, HS decomposition)[11]. For every matrix $A \in \mathbb{C}^{n\times n}$ with having rank r, we can write the matrix A as

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{2.5}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \operatorname{diag}(\sigma I_{r_1}, \sigma I_{r_2}, \cdots, \sigma I_{r_s})$ is the diagonal matrix with elements equal to singular values of A such that $\sigma_1 > \sigma_2 > \cdots > \sigma_s > 0$, $r_1 + r_2 + \cdots + r_s = r$. The blocks $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ are related with $KK^* + LL^* = I_r$.

Proposition 2.4 gives representations of the G-Drazin inverse based on the HS decomposition of A.

Proposition 2.4. *Consider the matrix A as defined in* (2.5)*. Then the GD inverses of A are of the form*

$$A^{\mathrm{GD}} = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^*,$$

where

$$\Sigma K X_1 + \Sigma L X_3 = I_r, \quad X_1(\Sigma K)^k = (\Sigma K)^{k-1}, \quad X_3(\Sigma K)^{k-1} = 0$$

and

$$(\Sigma K)^{k+1} X_2 + (\Sigma K)^k \Sigma L X_4 = (\Sigma K)^{k-1} \Sigma L.$$

Proposition 2.5 establishes the CEP inverse of A utilizing the HS decomposition.

Proposition 2.5. [8, Theorem 3.2] *Consider the matrix A as defined in* (2.5)*. Then $A^{\textcircled{\dagger}}$ is of the form*

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{bmatrix} U^*, \text{ where } ind(\Sigma K) = k-1.$$

The following result follows immediately in view of propositions 2.4 and 2.5.

Theorem 2.9. *Consider the matrix A as defined in* (2.5)*. Then CEPGD inverses of A are of the form*

$$A^{\textcircled{\dagger},\mathrm{GD}} = U \begin{bmatrix} (\Sigma K)^{\textcircled{\dagger}} & (\Sigma K)^{\textcircled{\dagger}}(\Sigma K X_2 + \Sigma L X_4) \\ 0 & 0 \end{bmatrix} U^*,$$

where

$$\Sigma K X_1 + \Sigma L X_3 = I_r, \quad X_1(\Sigma K)^k = (\Sigma K)^{k-1}, \quad X_3(\Sigma K)^{k-1} = 0$$

and

$$(\Sigma K)^{k+1} X_2 + (\Sigma K)^k \Sigma L X_4 = (\Sigma K)^{k-1} \Sigma L.$$

3. Binary relation on CEPGD inverses

It is well known that a reflexive and transitive binary relation on a non-empty set is a pre-order [23]. In addition, if the relation is also anti-symmetric, it is termed as a partial order.

Definition 3.1. [23, Definition 4.2.1] *Let $A, B \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = 1$*. Then A is below B under the sharp order $A \leq^{\#} B$ if there exist commuting g-inverses A^- and $A^=$ such that $AA^- = BA^-$ and $A^=A = A^=B$.*

A few examples of these relations are given below. The matrices $A,\ B \in \mathbb{C}^{n\times n}$ are assumed.

- Star partial order [23, Page 2]:
  We say $A \leq^* B$ if $A^\dagger A = A^\dagger$ and $AA^\dagger = BA^\dagger$.

- Left sharp partial order [23, Definition 6.3.1]:
If $\text{ind}(A) \leq 1$, we say $A \,\#\!\leq\, B$ if $A^2 = AB$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

- Right sharp partial order [23, Definition 6.3.1]:
If $\text{ind}(A) \leq 1$, we say $A \leq\!\# \, B$ if $A^2 = BA$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$.

- Core partial order [1]:
If $\text{ind}(A) = 1$, we say $A \leq^{\textcircled{\#}} B$ if $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$ and $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$.

- Core-EP pre-order [37]:
We say $A \leq^{\textcircled{\dagger}} B$ if $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$ and $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.

- Drazin pre-order [23, Page-118]:
We say $A \leq^{\mathrm{D}} B$ if $A^{\mathrm{D}}A = A^{\mathrm{D}}B$ and $AA^{\mathrm{D}} = BA^{\mathrm{D}}$.

We now establish a binary relation based on the CEPGD inverse, and then utilizing the definition, we characterize the relationship in terms of partial order when we consider the matrices of at most index 1.

Definition 3.2. *For $A, B \in \mathbb{C}^{n\times n}$, we will say that A is below B under the relation $\leq^{\textcircled{\dagger},\mathrm{GD}}$ if $A^{\textcircled{\dagger},\mathrm{GD}}A = A^{\textcircled{\dagger},\mathrm{GD}}B$ and $AA^{\textcircled{\dagger},\mathrm{GD}} = BA^{\textcircled{\dagger},\mathrm{GD}}$ for a fixed $A^{\mathrm{GD}} \in A\{GD\}$. Such a relation is termed as $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$.*

Clearly, the relation $\leq^{\textcircled{\dagger},\mathrm{GD}}$ is reflexive but need not be transitive, as shown in the below example.

Example 3.1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we can find that

$$A^{\mathrm{GD}} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B^{\mathrm{GD}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \; A^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be calculated

$$AA^{\textcircled{\dagger},\mathrm{GD}} = BA^{\textcircled{\dagger},\mathrm{GD}}, \quad A^{\textcircled{\dagger},\mathrm{GD}}A = A^{\textcircled{\dagger},\mathrm{GD}}B,$$

as well as

$$BB^{\textcircled{\dagger},\mathrm{GD}} = CB^{\textcircled{\dagger},\mathrm{GD}}, \quad B^{\textcircled{\dagger},\mathrm{GD}}B = B^{\textcircled{\dagger},\mathrm{GD}}C.$$

It is observable that $AA^{\textcircled{\dagger},\mathrm{GD}} = CA^{\textcircled{\dagger},\mathrm{GD}}$ but

$$A^{\textcircled{\dagger},\mathrm{GD}}C = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq A^{\textcircled{\dagger},\mathrm{GD}}A.$$

Thus, the conclusion is that both the relations $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$ and $B \leq^{\textcircled{\dagger},\mathrm{GD}} C$ hold, but $A \not\leq^{\textcircled{\dagger},\mathrm{GD}} C$.

Proposition 3.1. *The subsequent statements are mutually equivalent for $A, B \in \mathbb{C}^{n\times n}$:*

(i) $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$.

(ii) $AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger},\mathrm{GD}}B$.

(iii) $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger},\mathrm{GD}}B$ *and* $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.

Proof. (i)⇒(ii) Let $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$. Then

$$\begin{aligned} AA^{\textcircled{\dagger}}A &= AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}}A \\ &= BA^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A \\ &= BA^{\textcircled{\dagger}}A, \end{aligned}$$

and

$$AA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}A = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}B = AA^{\textcircled{\dagger},\mathrm{GD}}B.$$

(ii)⇒(iii) Let $AA^{\textcircled{\dagger}}A = BA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger},\mathrm{GD}}B$. Then

$$A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}A = A^{\textcircled{\dagger}}AA^{\mathrm{GD}}B = A^{\textcircled{\dagger},\mathrm{GD}}B$$

and

$$\begin{aligned} AA^{\textcircled{\dagger}} &= AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}} \\ &= BA^{\textcircled{\dagger}}AA^{\mathrm{GD}}AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}. \end{aligned}$$

(iii)⇒(i) Assume (iii) holds. Then it follows

$$A^{\textcircled{\dagger},\mathrm{GD}}A = A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger},\mathrm{GD}}B$$

in conjuction with

$$AA^{\textcircled{\dagger},\mathrm{GD}} = AA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = BA^{\textcircled{\dagger}}AA^{\mathrm{GD}} = BA^{\textcircled{\dagger},\mathrm{GD}}.$$

□

Theorem 3.1. *Assume that $A \in \mathbb{C}^{n\times n}$ is represented by* (2.2)*. In addition, if $B \in \mathbb{C}^{n\times n}$ the subsequent statements are equivalent:*

(i) $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$.

(ii) $B = U \begin{bmatrix} T_1 & T_2 - (T_1X_2 + T_2N^-)B_4 \\ 0 & B_4 \end{bmatrix} U^*$.

Proof. (i)⇒(ii) Let $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$ and consider $B = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^*$, where B_i $(i = 1, 2, 3, 4)$ are arbitrary. By comparing $AA^{\textcircled{\dagger},\mathrm{GD}} = BA$, we obtain

$$B_1 = T_1,\ B_3 = 0.$$

Applying $A^{\textcircled{\dagger},\mathrm{GD}}A = A^{\textcircled{\dagger},\mathrm{GD}}B$, we get

$$T_1^{-1}T_2 + (X_2 + T_1^{-1}T_2N^-)N = T_1^{-1}B_2 + (X_2 + T_1^{-1}T_2N^-)B_4. \tag{3.1}$$

Using $(T_1X_2+T_2N^-)N = 0$ (see Theorem 2.8) and the equality (3.1), we have $B_2 = T_2-(T_1X_2+T_2N^-)B_4$.
(ii)⇒(i) It follows by direct verification. □

Lemma 3.1. *Let A, $B \in \mathbb{C}^{n\times n}$ and assume $ind(A) \leq 1$. For a fixed $A^{\mathrm{GD}} \in A\{GD\}$ satisfying $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$, the following statements are valid:*

(i) *$A\# \leq B$, where $\# \leq$ is the left sharp order.*

(ii) *$A \leq \#B$, where $\leq \#$ is the right sharp order.*

Proof. For $\mathrm{ind}(A) = 0$, the result is trivial. Let $\mathrm{ind}(A) = 1 = k$. Thus $A^{\textcircled{\dagger}} = A^{\textcircled{\#}}$. Further, from $A^2A^{\mathrm{GD}} = A = A^{\mathrm{GD}}A^2$, we obtain $AA^{\mathrm{GD}} = A^{\mathrm{GD}}A$. Now if $A \leq^{\textcircled{\dagger},\mathrm{GD}} B$, then

$$AA^{\mathrm{GD}} = BA^{\textcircled{\#}}AA^{\mathrm{GD}} \text{ and } A^{\textcircled{\#}}A = A^{\textcircled{\#}}AA^{\mathrm{GD}}B.$$

(i) Using these properties, it is derived

$$A^2 = A^2 A^{\textcircled{\#}} A = A^2 A^{\textcircled{\#}} A A^{\mathrm{GD}} B = A^2 A^{\mathrm{GD}} B = AB.$$

The condition $R(A) \subseteq R(B)$ follows from $A = AA^{\mathrm{GD}} A = BA^{\textcircled{\#}} AA^{\mathrm{GD}} A$. Hence by [23, Definition 6.3.1], we conclude $A\# \leq B$.

(ii) It can be verified from the below expressions:

$$A^* = (AA^{\textcircled{\#}} A)^* = (A^{\textcircled{\#}} A)^* A^* = B^*(AA^{\textcircled{\#}} AA^{\mathrm{GD}})^*$$

and

$$A^2 = AA^{\mathrm{GD}} A^2 = BA^{\textcircled{\#}} AA^{\mathrm{GD}} A^2 = BA^{\textcircled{\#}} A^2 = BA.$$

□

Theorem 3.2. *Assume $A, B \in \mathbb{C}^{n\times n}$ and ind$(A) \leq 1$. Under these conditions, it follows*

$$A \leq^{\#} B \Longleftrightarrow A \leq^{\textcircled{\dagger},\mathrm{GD}} B,$$

for some $A^{\mathrm{GD}} \in A\{GD\}$.

Proof. Let $A \leq^{\#} B$. Then by [23, Theorem 4.2.5], we have $AA^{\#} = BA^{\#}$ and $A^{\#}A = A^{\#}B$. Now

$$\begin{aligned} AA^{\textcircled{\dagger},\mathrm{GD}} &= AA^{\mathrm{GD}} = AA^{\#} AA^{\mathrm{GD}} = BA^{\#} AA^{\mathrm{GD}} \\ &= BA^{\#} A^2 (A^{\mathrm{GD}})^2 = BA(A^{\mathrm{GD}})^2 \\ &= BA^{\textcircled{\#}} A^2 (A^{\mathrm{GD}})^2 = BA^{\textcircled{\#}} AA^{\mathrm{GD}} = BA^{\textcircled{\dagger},\mathrm{GD}}, \end{aligned}$$

and

$$\begin{aligned} A^{\textcircled{\dagger},\mathrm{GD}} A &= A^{\textcircled{\#}} A = A^{\textcircled{\#}} AA^{\#} A \\ &= A^{\textcircled{\#}} AA^{\#} B = A^{\textcircled{\#}} A^{\mathrm{GD}} A^2 A^{\#} B \\ &= A^{\textcircled{\#}} A^{\mathrm{GD}} AB = A^{\textcircled{\#}} AA^{\mathrm{GD}} B = A^{\textcircled{\dagger},\mathrm{GD}} B. \end{aligned}$$

Hence, by Lemma 3.1, the proof is complete. □

Remark 3.1. *The relation $\leq^{\textcircled{\dagger},\mathrm{GD}}$ is a partial order on the set $\mathcal{PO} = \{A, B \in \mathbb{C}^{n\times n} : \mathrm{ind}(A) = \mathrm{ind}(B) \leq 1\}$.*

4. Conclusion

A novel class of outer generalized inverses, termed as CEPGD inverse, is introduced as a proper composition of the core-EP and the G-Drazin inverse. A few properties and computationally efficient representations of the CEPGD inverses are presented and investigated. The image and nullity of CEPGD inverses are considered. The representations of CEPGD inverses based on the core-nilpotent decomposition and the Hartwig-Spindelböck decomposition are established. A binary relation induced by these inverses is introduced along with some derived properties.

Some encouraging subjects for future investigation are mentioned as follows:
- development of iterations for computing the CEPGD inverses;
- perturbations, limit, and continuity of the CEPGD inverses;
- studying of CEPGD inverses for tensors;
- investigation of CEPGD inverses for Hilbert spaces operators.

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