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ORIGINAL PAPER

Application of *m*-weak group inverse in solving optimization problems

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Abstract
Several new expressions are proved for the m-weak group inverse. An effective algorithm for computing m-weak group inverse in terms of the QR decomposition is proposed. Applying the m-weak group inverse, we present the uniquely determined solution to the restricted minimization problem in the Frobenius norm: $\min \|A^{m+1}X - A^m B\|_F$ provided that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, where $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$.

Keywords Core-EP inverse · Drazin inverse · Constrained matrix optimization problem

Mathematics Subject Classification 15A09 · 15A24 · 65F05

1 Introduction

Standardly, for $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices, let $\mathrm{rank}(A)$, A^*, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, be its rank, conjugate-transpose, null space and range space. As usual, $\mathbb{C}_r^{m\times n} = \left\{A \in \mathbb{C}^{m\times n} \mid \mathrm{rank}(A) = r\right\}$. The symbol P_U represents the orthogonal projector onto a subspace U.

We firstly introduce a few of significant generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse of A is the unique matrix $X = A^\dagger \in \mathbb{C}^{n\times m}$ satisfying (see [3])

$$XAX = X, \qquad AXA = A, \qquad (AX)^* = AX, \qquad (XA)^* = XA.$$

Recall that, for $A \in \mathbb{C}^{m\times n}$, the set of all outer inverses (or also called $\{2\}$-inverses) is defined by $A\{2\} = \{X \in \mathbb{C}^{n\times m} \mid XAX = X\}$, and the set of all outer inverses of rank s is

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denoted by $A\{2\}_s$. For a subspace T of $\mathbb{C}^n$ with dimension $p \le \mathrm{rank}(A)$ and a subspace S of $\mathbb{C}^m$ with dimension $m-p$, an outer inverse X of $A \in \mathbb{C}^{m\times n}$ with the range space $\mathcal{R}(X)=T$ and null space $\mathcal{N}(X)=S$ is unique (if it exists) and it is denoted by $A^{(2)}_{T,S}$ [3].

For $A \in \mathbb{C}^{n\times n}$ and $k=\mathrm{ind}(A)$, where $\mathrm{ind}(A)$ represents the index of A (i.e. the smallest nonnegative integer k such that $\mathrm{rank}(A^k)=\mathrm{rank}(A^{k+1})$), there exists its Drazin inverse [3], i.e. the unique $X=A^{\mathrm{D}} \in \mathbb{C}^{n\times n}$ such that

$$XAX=X, \quad A^{k+1}X=A^k, \quad AX=XA.$$

In the special case $\mathrm{ind}(A)=1$, A^{D} becomes the group inverse $A^{\#}$ of A.

Let $A \in \mathbb{C}^{n\times n}$ and $k=\mathrm{ind}(A)$. The core-EP inverse of A is the unique matrix $X=A^{\textcircled{\dagger}} \in \mathbb{C}^{n\times n}$ which satisfies [17]

$$XAX=X, \quad \mathcal{R}(A^k)=\mathcal{R}(X)=\mathcal{R}(A^*).$$

A helpful expression was presented in [8] for calculating the core-EP inverse as

$$A^{\textcircled{\dagger}}=A^{\mathrm{D}}A^k(A^k)^{\dagger}=A^k\left(A^{k+1}\right)^{\dagger}.$$

If $\mathrm{ind}(A)=1$, $A^{\textcircled{\#}}=A^{\textcircled{\dagger}}=A^{\#}AA^{\dagger}$ is the core inverse of A [1]. Many significant results related to the core-EP inverse can be seen in [2, 5, 10, 12, 14, 18, 32].

The notion of the group inverse was extended modifying or omitting equations which are used in its definition. In addition to the theoretical significance, the group inverse is very important in applications such as in Markov chains, solving differential and difference equations [3, 4].

The weak group inverse (or WGI) is a kind of generalization of the group inverse given in [24]. In particular, the WGI of $A \in \mathbb{C}^{n\times n}$ is expressed as

$$A^{\textcircled{w}}=(A^{\textcircled{\dagger}})^2A$$

and it is the unique solution to the system of matrix equations

$$AX=A^{\textcircled{\dagger}}A, \quad AX^2=X.$$

When $\mathrm{ind}(A)=1$, the WGI reduces to the group inverse. More results about the WGI can be found in [7, 11, 16, 25, 31, 33].

The m-weak group inverse (or m-WGI) was defined in [34] as an extension of the concept of the WGI. For $m \in \mathbb{N}$, the m-WGI of $A \in \mathbb{C}^{n\times n}$ is defined by

$$A^{\textcircled{w}_m}=(A^{\textcircled{\dagger}})^{m+1}A^m$$

and it presents the unique matrix X which represents the solution of matrix equations

$$AX=(A^{\textcircled{\dagger}})^mA^m, \quad AX^2=X.$$

In the case $m=1$, the m-WGI becomes the WGI. When $m=2$, notice that the m-WGI is equal to the generalized group (or GG) inverse defined in [6] by

$$A^{\textcircled{w}_2}=(A^{\textcircled{\dagger}})^3A^2.$$

If $\mathrm{ind}(A) \le m$, the m-WGI reduced to the Drazin inverse. Some properties of the m-WGI were presented in [9, 15].

Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A)=k$. For $b \in \mathcal{R}(A^k)$, it is know that $x=A^{\mathrm{D}}b$ represents the unique Drazin inverse solution to the constrained linear system [4]

$$Ax=b, \qquad x \in \mathcal{R}(A^k).$$

Especially, when $\mathrm{ind}(A) = 1$ and $b \in \mathcal{R}(A)$, $x = A^{\#}b (= A^{\textcircled{\#}}b)$ is the unique solution to $Ax = b$. In the case $b \in \mathbb{C}^n$, i.e. omitting the hypothesis $b \in \mathcal{R}(A)$, it was proved that $x = A^{\textcircled{\#}}b$ presents the unique solution to the restricted matrix minimization problem in the Frobenius norm [27]:

$$\min \|Ax - b\|_F \quad \text{subjectto} \quad x \in \mathcal{R}(A).$$

Extending this problem for complex matrices with index one considered in [27] to complex matrices with arbitrary index, one can see that $x = A^{\textcircled{\dagger}}b$ is uniquely determined solution to the general constrained matrix minimization problem in the Euclidean norm [13]

$$\min \|Ax - b\|_2 \quad \text{subjectto} \quad x \in \mathcal{R}(A^k), \tag{1.1}$$

where $b \in \mathbb{C}^n$, $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. Notice that the condition $b \in \mathcal{R}(A^k)$, given in [4], is omitted.

The matrix minimization problem in the Frobenius norm [26]

$$\min \|A^2X - AB\|_F \quad \text{subjectto} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k), \tag{1.2}$$

where $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, was solved applying the WGI and $X = A^{\textcircled{w}}B$ is the unique solution to (1.2). Recall that, by $\mathrm{rank}(A^2) \le \mathrm{rank}(A)$, the restricted equation $A^2X = AB$ provided that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, is not always consistent, which is the reason why the least square solution of it is investigated.

Motivated by previous researches about solvability of mentioned optimization problems, our main goal is to study the most general minimization problem which will extend and recover the known results. Precisely, our aim is to solve the constrained optimization problem in the Frobenius norm:

$$\min \|A^{m+1}X - A^mB\|_F \quad \text{subjectto} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k), \tag{1.3}$$

where $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. The detailed explanation of our results follows.

(1) We prove that the problem (1.3) possesses a unique solution and it is provided by the m-WGI. Notice that the problem proposed in [26], namely the problem (1.2), is a particular case of our optimization problem for $m = 1$. So, we recover a wider class of minimization problems.
(2) Some special cases of the problem (1.3) are considered and solved.
(3) New formulae for computing the m-WGI are presented using the full-rank factorization of A^k.
(4) More expressions of the m-WGI are established applying a matrix V such that the kernel of V^* is equal to the range space of A^k.
(5) The algorithm for verification of theoretical results is developed and implemented.

The organization of our sections follows. In Sect. 2, new expressions for the m-WGI are proposed. Section 3 involves investigations related to the solvability of the minimization problem (1.3) and its particular cases. Numerical examples are given in Sect. 4. Final section presents some concluding observations.

2 Representations of the *m*-weak group inverse

Let us assume $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ throughout this section. Firstly, we state the known decomposition of A presented in [23] and the corresponding representation for the m-weak group inverse proposed in [9].

Lemma 2.1 [23] *If* $\mathrm{rank}(A^k) = t$*, then*

$$A = U \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} U^*, \tag{2.1}$$

for a unitary $U \in \mathbb{C}^{n\times n}$*, a nonsingular upper-triangular* $A_1 \in \mathbb{C}^{t\times t}$ *and a nilpotent* $A_3 \in \mathbb{C}^{(n-t)\times(n-t)}$ *of index k. Further, it follows, for* $m \in \mathbb{N}$, [9]*:*

$$A^{Ⓦ_m} = U \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & 0 \end{bmatrix} U^*. \tag{2.2}$$

Theorem 2.1 gives representations of the m-weak group inverse based on the full-rank decomposition of A^k.

Theorem 2.1 *For* $m \in \mathbb{N}$ *and a full-rank decomposition* $A^k = EF$*, it follows that* $E^*A^{m+1}E$ *is nonsingular and*

$$A^{Ⓦ_m} = E(E^*A^{m+1}E)^{-1}E^*A^m \tag{2.3}$$

$$= A^{(2)}_{\mathcal{R}(E),\mathcal{N}(E^*A^m)}. \tag{2.4}$$

Proof Assume that A is expressed by (2.1). Then

$$A^k = \left(U \begin{bmatrix} A_1^k \\ 0 \end{bmatrix}\right)\left(\begin{bmatrix} I_t & A_1^{-k} \sum_{i=0}^{k-1} A_1^{k-i-1} A_2 A_3^i \end{bmatrix} U^*\right) := PQ$$

is another full-rank factorization of A^k. Since $A^k = EF$ is a full-rank decomposition of A^k, then FE is nonsingular. For $G = QE$, from

$$t = \mathrm{rank}(A^k) = \mathrm{rank}(A^{2k}) = \mathrm{rank}(PGF) \le \mathrm{rank}(G) \le t,$$

we deduce that G is nonsingular. Notice that $Z := G(FE)^{-1}$ is nonsingular and

$$E = EFE(FE)^{-1} = PQE(FE)^{-1} = PG(FE)^{-1} = PZ = U \begin{bmatrix} A_1^k Z \\ 0 \end{bmatrix}.$$

Since

$$\begin{aligned} E^*A^{m+1}E &= \begin{bmatrix} Z^*(A_1^k)^* & 0 \end{bmatrix} U^*U \begin{bmatrix} A_1^{m+1} & \sum_{j=0}^{m-1} A_1^{j+1} A_2 A_3^{m-1-j} + A_2 A_3^m \\ 0 & A_3^{m+1} \end{bmatrix} U^*U \begin{bmatrix} A_1^k Z \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} Z^*(A_1^k)^* & 0 \end{bmatrix} \begin{bmatrix} A_1^{m+k+1} Z \\ 0 \end{bmatrix} \\ &= Z^*(A_1^k)^* A_1^{m+k+1} Z, \end{aligned}$$

it follows that $E^*A^{m+1}E$ is nonsingular. Using

$$E^*A^m = \begin{bmatrix} Z^*(A_1^k)^* & 0 \end{bmatrix} U^*U \begin{bmatrix} A_1^m & \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & A_3^m \end{bmatrix} U^*$$
$$= \begin{bmatrix} Z^*(A_1^k)^* A_1^m & Z^*(A_1^k)^* \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \end{bmatrix} U^*,$$

we obtain

$$\begin{aligned} E(E^*A^{m+1}E)^{-1}E^*A^m &= U \begin{bmatrix} A_1^k Z \\ 0 \end{bmatrix} (Z^*(A_1^k)^* A_1^{m+k+1} Z)^{-1} E^*A^m \\ &= U \begin{bmatrix} A_1^k Z \\ 0 \end{bmatrix} (A_1^k Z)^{-1} A_1^{-(m+1)} (Z^*(A_1^k)^*)^{-1} \\ &\quad \times \begin{bmatrix} Z^*(A_1^k)^* A_1^m & Z^*(A_1^k)^* \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \end{bmatrix} U^* \\ &= U \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & 0 \end{bmatrix} U^* \\ &= A^{ⓦ_m}. \end{aligned}$$

The representation $E\,(E^*A^mAE)^{-1}\,E^*A^m$ is another form of (2.3). Finally, (2.4) follows from the full-rank representation of outer inverses $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ with prescribed range space and kernel given by the Urquhart representation [22] in the form $B(CAB)^\dagger C$ and its extensions given in [20]. □

Remark 2.1 The representation (2.3) verified in Theorem 2.1 extends the representation

$$A^{ⓦ} = E\left(E^*A^2E\right)^{-1} E^*A$$

of the WG inverse given in [26, Theorem 5]. More precisely, the representation of the WG inverse given in [26, Theorem 5] is the particular case $m = 1$ of Theorem 2.1.

For $m \geq k$, the following formulae for the Drazin inverse are consequences of Theorem 2.1.

Corollary 2.1 *For $m \geq k$ and a full-rank decomposition $A^k = EF$, it follows*

$$\begin{aligned} A^{\mathrm{D}} &= E(E^*A^{m+1}E)^{-1}E^*A^m \\ &= A^{(2)}_{\mathcal{R}(E),\mathcal{N}(E^*A^m)}. \end{aligned}$$

In Corollary 2.2, we derive a full-rank factorization of A^k in terms of its specific QR decomposition. Then the statements in Corollary 2.2 follow from Theorem 2.1.

Corollary 2.2 *Assume that the matrix $A \in \mathbb{C}_r^{n\times n}$ satisfies $\mathrm{ind}(A) = k$. Suppose that A^k satisfies $\mathrm{rank}(A^k) = s$ and the QR factorization of A^k is of the form*

$$A^k = Q^*RP^*, \tag{2.5}$$

where P is appropriate $n \times n$ permutation matrix, $Q \in \mathbb{C}^{n\times n}$ is partitioned as

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \tag{2.6}$$

and $Q_1 \in \mathbb{C}^{s\times n}$ *fulfils* $Q_1^{-1} = Q_1^*$, *and* $R \in \mathbb{C}_s^{n\times n}$ *is upper trapezoidal. Let* R *be partitioned as*

$$R = \begin{bmatrix} R_{11} & R_{12} \\ O & O \end{bmatrix} = \begin{bmatrix} R_1 \\ O \end{bmatrix}, \tag{2.7}$$

where $R_1 \in \mathbb{C}^{s\times s}$ *is nonsingular.*

The subsequent claim are valid:

(a) $Q_1 A^{m+1} Q_1^*$ *is invertible;*
(b) $A^{ⓦ_m} = Q_1^*(Q_1 A^{m+1} Q_1^*)^{-1} Q_1 A^m = A^{(2)}_{\mathcal{R}(Q_1^*),\mathcal{N}(Q_1 A^m)}$;
(c) $A^{ⓦ_m} \in A\{2\}_s$.

Proof Since $A^k = Q_1^*(R_1 P^*)$ is the full-rank decomposition of A^k, the proof follows from Theorem 2.1. □

Algorithm 2.1 is aimed to computing $A^{ⓦ_m}$ and based on Corollary 2.2.

Algorithm 2.1 Computing $A^{ⓦ_m}$ using the QR decomposition of A^k.
(Algorithm QRm-WGI)

Require: The $n \times n$ matrix A.
1: Compute $k = \text{ind}(A)$.
2: Choose an integer $m < k$.
3: Compute the QR decomposition (2.5) of A^k, where Q and R are of the form (2.6) and (2.7), respectively.
4: Solve $Q_1 A^{m+1} Q_1^* X = Q_1 A^m$.
5: Return $A^{ⓦ_m} = Q_1^* X$.

Since the formula for $A^{ⓦ_m}$ given in Theorem 2.1 depends only on E from the full-rank decomposition $A^k = EF$, we present one more expression for $A^{ⓦ_m}$. Notice that Theorem 2.2 is a generalization of [26, Theorem 6].

Theorem 2.2 *Let* $S \in \mathbb{C}^{n\times n}$ *such that* $\mathcal{R}(S) = \mathcal{R}(A^k)$. *For* $m \in \mathbb{N}$, *the m-weak group inverse is defined by*

$$\begin{aligned} A^{ⓦ_m} &= S(S^* A^{m+1} S)^\dagger S^* A^m \\ &= A^{(2)}_{\mathcal{R}(S),\mathcal{N}(S^* A^m)}. \end{aligned}$$

Proof Suppose that $A^k = EF$ and $S = S_1 S_2$, respectively, are full-rank decompositions of A^k and S. The hypothesis $\mathcal{R}(S) = \mathcal{R}(A^k)$ implies that $S_1 = EZ$, for some invertible Z. So, $S = E(ZS_2)$ is a full-rank factorization of S. By Theorem 2.1, $E^* A^{m+1} E$ is nonsingular and

$$S^* A^{m+1} S = (E(ZS_2))^* A^{m+1} E(ZS_2) = (S_2^* Z^* E^* A^{m+1} E) Z S_2$$

is a full-rank factorization of $S^* A^{m+1} S$. Now,

$$\begin{aligned} S(S^* A^{m+1} S)^\dagger S^* A^m &= E(ZS_2)((S_2^* Z^* E^* A^{m+1} E) Z S_2)^\dagger S_2^* Z^* E^* A^m \\ &= E(ZS_2)(ZS_2)^*((ZS_2)(ZS_2)^*)^{-1} \\ &\quad \times ((S_2^* Z^* E^* A^{m+1} E)^* (S_2^* Z^* E^* A^{m+1} E))^{-1} (S_2^* Z^* E^* A^{m+1} E)^* \\ &\quad \times S_2^* Z^* E^* A^{m+1} E (E^* A^{m+1} E)^{-1} E^* A^m \\ &= E(E^* A^{m+1} E)^{-1} E^* A^m. \end{aligned}$$

The rest is clear by Theorem 2.1. □

As a consequence of Theorem 2.2, we obtain the next expressions for the Drazin inverse.

Corollary 2.3 *Let $S \in \mathbb{C}^{n\times n}$ such that $\mathcal{R}(S) = \mathcal{R}(A^k)$. For $m \geq k$, it follows*

$$\begin{aligned} A^{\mathrm{D}} &= S(S^*A^{m+1}S)^{\dagger}S^*A^m \\ &= A^{(2)}_{\mathcal{R}(S),\mathcal{N}(S^*A^m)}. \end{aligned}$$

Applying the formula for the core-EP inverse proposed in [13, Theorem 2.3], we get a novel expression of the m-WGI.

Theorem 2.3 *For $m \in \mathbb{N}$ and a matrix V such that $\mathcal{N}(V^*) = \mathcal{R}(A^k)$ (or $A^kV^* = 0$), the matrix $A^k(A^k)^*A + VV^*$ is nonsingular and*

$$\begin{aligned} A^{Ⓦ_m} &= \left(A^k(A^k)^*A + VV^*\right)^{-1} A^k(A^k)^*(A^{\mathrm{D}})^m A^k(A^k)^{\dagger}A^m \\ &= \left(A^k(A^k)^*A + VV^*\right)^{-1} A^k(A^k)^*A^k(A^{k+m})^{\dagger}A^m. \end{aligned}$$

Proof Using $A^{ⓣ} = A^{\mathrm{D}}A^k(A^k)^{\dagger}$, we can check that $(A^{ⓣ})^m = (A^{\mathrm{D}})^m A^k(A^k)^{\dagger}$. By [13, Theorem 2.3], it follows that $A^k(A^k)^*A + VV^*$ is nonsingular and $A^{ⓣ} = (A^k(A^k)^*A + VV^*)^{-1}A^k(A^k)^*$. Thus,

$$A^{Ⓦ_m} = (A^{ⓣ})^{m+1}A^m = \left(A^k(A^k)^*A + VV^*\right)^{-1} A^k(A^k)^*(A^{\mathrm{D}})^m A^k(A^k)^{\dagger}A^m.$$

The second equality is clear by $A^k(A^k)^{\dagger} = P_{\mathcal{R}(A^k)} = P_{\mathcal{R}(A^{k+m})} = A^{k+m}(A^{k+m})^{\dagger}$. □

Based on [13, Theorem 2.4], we have one more representation for $A^{Ⓦ_m}$.

Theorem 2.4 *For $m \in \mathbb{N}$ and a matrix V such that $\mathcal{N}(V^*) = \mathcal{R}(A^k)$, the matrix $A^{k+1} + VV^*$ is nonsingular and*

$$\begin{aligned} A^{Ⓦ_m} &= \left(A^k + (A^{\mathrm{D}})^{m+1}A^k(A^k)^{\dagger}A^mVV^*\right)\left(A^{k+1} + VV^*\right)^{-1} \\ &= \left(A^k + A^k(A^{k+m+1})^{\dagger}A^mVV^*\right)\left(A^{k+1} + VV^*\right)^{-1}. \end{aligned} \tag{2.8}$$

Proof According to [13, Theorem 2.4], note that $A^{k+1} + VV^*$ is nonsingular. Hence,

$$\begin{aligned} A^{Ⓦ_m}(A^{k+1} + VV^*) &= (A^{ⓣ})^{m+1}A^m(A^{k+1} + VV^*) \\ &= (A^{\mathrm{D}})^{m+1}A^k(A^k)^{\dagger}A^m(A^{k+1} + VV^*) \\ &= (A^{\mathrm{D}})^{m+1}A^{k+m+1} + (A^{\mathrm{D}})^{m+1}A^k(A^k)^{\dagger}A^mVV^* \\ &= A^k + (A^{\mathrm{D}})^{m+1}A^k(A^k)^{\dagger}A^mVV^* \end{aligned}$$

yields the validity of (2.8). □

Notice that, the formulae for $A^{Ⓦ_m}$ proved in Theorem 2.3 and Theorem 2.4, are satisfied for arbitrary $l \geq k$ instead of k.

Especially, for $m = 1$ in Theorem 2.3 and Theorem 2.4, we get new expressions for the weak group inverse.

Corollary 2.4 *For a matrix V such that $\mathcal{N}(V^*) = \mathcal{R}(A^k)$, we have*

$$A^{ⓦ} = \left(A^k(A^k)^*A + VV^*\right)^{-1} A^k(A^k)^*A^k(A^{k+1})^\dagger A$$
$$= \left(A^k + A^k(A^{k+2})^\dagger A\right)\left(A^{k+1} + VV^*\right)^{-1}.$$

Theorem 2.3 and Theorem 2.4 also implies representations for the Drazin inverse when $m = k$.

Corollary 2.5 *For a matrix V such that $\mathcal{N}(V^*) = \mathcal{R}(A^k)$, the matrices $A^k(A^k)^*A + VV^*$ and $A^{k+1} + VV^*$ are nonsingular and*

$$A^{ⓦ_m} = \left(A^k(A^k)^*A + VV^*\right)^{-1} A^k(A^k)^*A^k(A^{2k})^\dagger A^k$$
$$= \left(A^k + A^k(A^{2k+1})^\dagger A^k VV^*\right)\left(A^{k+1} + VV^*\right)^{-1}.$$

3 Solving (1.3) in terms of the *m*-weak group inverse

Using the m-weak group inverse, we determine the unique solution to the constrained matrix minimization problem (1.3).

Theorem 3.1 *For $m \in \mathbb{N}$, the optimization problem (1.3) has the unique solution of the form*

$$X = A^{ⓦ_m}B.$$

Proof **(a)** Consider $m < \mathrm{ind}(A)$. The condition $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$ gives $X = A^kY$ for some $Y \in \mathbb{C}^{n\times q}$. Suppose that A is given by (2.1) as well as

$$U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad U^*Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad B_1, Y_1 \in \mathbb{C}^{t\times m}.$$

Notice that X is a solution to the problem (1.3) if and only if Y is a solution to

$$\min \|A^{m+k+1}Y - A^mB\|_F^2.$$

Using

$$A^k = U\begin{bmatrix} A_1^k & \sum_{i=0}^{k-1} A_1^{k-i-1}A_2A_3^i \\ 0 & 0 \end{bmatrix}U^* \quad \text{and} \quad A^m = U\begin{bmatrix} A_1^m & \sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j} \\ 0 & A_3^m \end{bmatrix}U^*,$$

we obtain

$$A^{m+k+1}Y = U\begin{bmatrix} A_1^{m+1} & \sum_{j=0}^{m-1} A_1^{j+1}A_2A_3^{m-1-j} + A_2A_3^m \\ 0 & A_3^{m+1} \end{bmatrix}U^*A^kY$$
$$= U\begin{bmatrix} A_1^{m+k+1} & \sum_{i=0}^{k-1} A_1^{m+k-i}A_2A_3^i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$
$$= U\begin{bmatrix} A_1^{m+k+1}Y_1 + \sum_{i=0}^{k-1} A_1^{m+k-i}A_2A_3^iY_2 \\ 0 \end{bmatrix}$$

and

$$A^m B = U \begin{bmatrix} A_1^m & \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & A_3^m \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$
$$= U \begin{bmatrix} A_1^m B_1 + \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \\ A_3^m B_2 \end{bmatrix}.$$

Thus,

$$\|A^{m+k+1}Y - A^m B\|_F^2 =$$
$$= \left\| \begin{bmatrix} A_1^{m+k+1} Y_1 + \sum_{i=0}^{k-1} A_1^{m+k-i} A_2 A_3^i Y_2 - A_1^m B_1 - \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \\ -A_3^m B_2 \end{bmatrix} \right\|_F^2$$
$$= \left\| A_1^{m+k+1} Y_1 + \sum_{i=0}^{k-1} A_1^{m+k-i} A_2 A_3^i Y_2 - A_1^m B_1 - \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \right\|_F^2 + \|A_3^m B_2\|_F^2,$$

which implies

$$\min_Y \|A^{m+k+1}Y - A^m B\|_F^2 = \|A_3^m B_2\|_F^2$$

for arbitrary $Y_2 \in \mathbb{C}^{(n-t)\times q}$ and

$$Y_1 = -A_1^{-(m+k+1)} \left(\sum_{i=0}^{k-1} A_1^{m+k-i} A_2 A_3^i Y_2 - A_1^m B_1 - \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \right).$$

The formula (2.2) yields

$$A^{\textcircled{w}_m} B = U \begin{bmatrix} A_1^{-1} B_1 + A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \\ 0 \end{bmatrix}.$$

Therefore,

$$X = A^k Y = U \begin{bmatrix} A_1^k Y_1 + \sum_{i=0}^{k-1} A_1^{k-i-1} A_2 A_3^i Y_2 \\ 0 \end{bmatrix}$$
$$= U \begin{bmatrix} -A_1^{-(m+1)} \left(\sum_{i=0}^{k-1} A_1^{m+k-i} A_2 A_3^i Y_2 - A_1^m B_1 - \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \right) + \sum_{i=0}^{k-1} A_1^{k-i-1} A_2 A_3^i Y_2 \\ 0 \end{bmatrix}$$
$$= U \begin{bmatrix} A_1^{-1} B_1 + A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_2 \\ 0 \end{bmatrix}$$
$$= A^{\textcircled{w}_m} B$$

presents uniquely determined solution to the problem (1.3).

(b) In the case $m \geq \mathrm{ind}(A)$ the proof follows from [19, Theorem 2.1], which claims that in the case $k = \mathrm{ind}(A)$ the Drazin inverse solution $A^{\mathrm{D}}B$ is the unique solution to the restricted equation

$$AX = B, \ \ \mathcal{R}(X) \subseteq \mathcal{R}(A^k), \ \mathcal{R}(B) \subseteq \mathcal{R}(A^k).$$

It is sufficient to observe $X = A^k Y$, $Y \in \mathbb{C}^{n\times n}$ and $B = A^k D$, $D \in \mathbb{C}^{n\times n}$. □

Remark that [26, Theorem 1] is a special case of Theorem 3.1 for $m = 1$.

In the case $B = I_n$ in Theorem 3.1, $A^{\circledS m}$ is uniquely determined solution to the next minimization problem.

Corollary 3.1 *For* $m < \mathrm{ind}(A)$*, the optimization problem*

$$\min \|A^{m+1}X - A^m\|_F \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k)$$

has the unique solution of the form

$$X = A^{\circledS m}.$$

The following result is a consequence of Theorem 3.1.

Corollary 3.2 *For* $m < \mathrm{ind}(A)$ *and* $b \in \mathbb{C}^n$*, the optimization problem*

$$\min \|A^{m+1}x - A^m b\|_2 \quad \text{subject to} \quad x \in \mathcal{R}(A^k)$$

has the unique solution of the form

$$x = A^{\circledS m} b.$$

The result of Corollary 3.2 can be extended to known results about the Drazin inverse solution for choices $m \geq k = \mathrm{ind}(A)$.

Proposition 3.1 [29, Theorem 3.1] *For* $A \in \mathbb{C}^{n\times n}$ *of index* $\mathrm{ind}(A) = k$*, the Drazin inverse solution* $A^{\mathrm{D}}b$ *is the unique solution of* $A^{k+1}x = A^k b, \ \ x \in \mathcal{R}(A^k)$.

4 Numerical examples

The QR decomposition required in Algorithm 2.1 and used in subsequent examples is generated adopting the results from [28, Theorem 3.3.11] and [21]. A practical implementation is developed using the function `QRDecomposition` from Wolfram *Mathematica* [30].

The following notation will be useful in numerical tests. The identity (resp. zero) $\ell \times \ell$ matrix will be denoted by $\mathbf{I}_\ell$ (resp. $\mathbf{0}_\ell$). Denote by $\mathbf{D}_\ell^p(\mathbf{v})$, $p \geq 1$, the $\ell \times \ell$ matrix with its pth leading diagonal parallel filled by the entries of the vector $\mathbf{v} \in \mathbb{C}^{\ell-p}$, and 0 in all other entries. Next, consider the simpler notation $\mathbf{D}_\ell^p = \mathbf{D}_\ell^p(\mathbf{1})$, $p \geq 1$, where $\mathbf{1} = \{1, \ldots, 1\} \in \mathbb{C}^{\ell-p}$.

Suitable test examples are matrices with a comparatively great index relative to their dimensions. That is why the class of test matrices of the form

$$\left\{ \begin{pmatrix} \mathbf{I}_\ell & C_1\mathbf{I}_\ell \\ \mathbf{0}_\ell & C_2\mathbf{D}_\ell^1(\mathbf{v}) \end{pmatrix}, \ \ \ell > 0 \right\}, \ \ C_1, C_2 \in \mathbb{C} \tag{4.1}$$

was tested in performed numerical experiments.

Example 4.1 The test matrices in this example are derived using $\ell = 4$ and $C_1 = C_2 = 1$ from the test set (4.1). Our intention is to perform numerical experiments on integer matrices using exact computation.

Consider the matrix

$$A = \begin{pmatrix} \mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{0}_4 & \mathbf{D}_4^1 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since $\{\mathrm{rank}(A), \mathrm{rank}(A^2), \mathrm{rank}(A^3), \mathrm{rank}(A^4), \mathrm{rank}(A^5)\} = \{7, 6, 5, 4, 4\}$, it follows $k = \mathrm{ind}(A) = 4$.

(a) In the first part of this example, we calculate the Drazin inverse, the core-EP inverse and m-WGI inverses according to their definitions. The Drazin inverse of A is computed using

$$A^{\mathrm{D}} = A^k \left(A^{2k+1}\right)^{\dagger} A^k = A^4 \left(A^9\right)^{\dagger} A^4 = \begin{pmatrix} \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{N}_4^{[3]} \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

and the core-EP inverse of A is

$$A^{\textcircled{\dagger}} = A^k \left(A^{k+1}\right)^{\dagger} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The WGI (or 1-WGI) inverse of A is given by

$$A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A = \begin{pmatrix} \mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

the 2-WGI (or GG) inverse of A is equal to

$$A^{\textcircled{w}2} = (A^{\textcircled{\dagger}})^3 A^2 = \begin{pmatrix} \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{N}_4^{[1]} \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

and the 3-WGI inverse of A is

$$A^{\textcircled{w}3} = (A^{\textcircled{\dagger}})^4 A^3 = \begin{pmatrix} \mathbf{I}_4 & \mathbf{I}_4 + \mathbf{N}_4^{[2]} \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Additionally, $A^{\textcircled{w}m} = A^{\mathrm{D}}$ is checked for each $m \geq \mathrm{ind}(A)$.
(b) In the general case, the matrix

$$A = \begin{pmatrix} \mathbf{I}_\ell & \mathbf{I}_\ell \\ \mathbf{0}_\ell & \mathbf{D}_\ell^1 \end{pmatrix}, \quad \ell \geq 2$$

is of index $\mathrm{ind}(A) = \ell$. The Drazin inverse and the core-EP inverse of A are equal to

$$A^{\mathrm{D}} = \begin{pmatrix} \mathbf{I}_\ell & \mathbf{I}_\ell + \mathbf{N}_\ell^{[\ell-1]} \\ \mathbf{0}_\ell & \mathbf{0}_\ell \end{pmatrix}, \quad A^{\textcircled{\dagger}} = A^k \left(A^{k+1}\right)^{\dagger} = \begin{pmatrix} \mathbf{I}_\ell & \mathbf{0}_\ell \\ \mathbf{0}_\ell & \mathbf{0}_\ell \end{pmatrix}.$$

Further calculation gives

$$A^{\textcircled{w}m} = (A^{\textcircled{\dagger}})^{m+1} A^m = \begin{cases} \begin{pmatrix} \mathbf{I}_\ell & \mathbf{I}_\ell + \mathbf{N}_\ell^{[m-1]} \\ \mathbf{0}_\ell & \mathbf{0}_\ell \end{pmatrix}, & m = 1, \ldots, \mathrm{ind}(A) - 1, \\ A^{\mathrm{D}} = \begin{pmatrix} \mathbf{I}_\ell & \mathbf{I}_\ell + \mathbf{N}_\ell^{[\ell-1]} \\ \mathbf{0}_\ell & \mathbf{0}_\ell \end{pmatrix}, & m \geq \mathrm{ind}(A). \end{cases}$$

(c) In this part of the example, we generate the class of m-WGI inverses according to statements in Corollary 2.2. Since $\mathrm{ind}(A) = 4$, is a full-rank factorization of A^4 is required. The QR factorization $A^4 = Q^* R P^{\mathrm{T}}$ is defined by the command `{Q,R,P}=QRDecomposition[MatrixPower[A,4]]` whose output is the ordered triple

$$\{Q, R, P\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, I_8 \right\}.$$

The following equalities are verified to check Corollary 2.2:

$$Q^*(QA^{m+1}Q^*)^{-1}QA^m = A^{(2)}_{\mathcal{R}(Q^*),\mathcal{N}(QA^m)} = \begin{cases} (A^{\textcircled{\dagger}})^{m+1}A^m, & m = 1,\dots,\operatorname{ind}(A)-1, \\ A^{\mathrm{D}}, & m \geq \operatorname{ind}(A) \end{cases} = A^{\textcircled{w}_m}.$$

(d) Consider the matrix

$$B = \begin{pmatrix} 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \end{pmatrix}$$

with intention to verify Theorem 3.1. The solution to the least-squares problem $\min\left\{\|A^2X - AB\|_F,\ \mathcal{R}(X) \subseteq \mathcal{R}(A^4)\right\}$ is given by

$$A^{\textcircled{w}}B = \begin{pmatrix} 3 & 2 & 3 & 1 & 1 \\ 4 & 2 & 4 & 4 & 2 \\ 0 & 2 & 0 & 2 & 2 \\ 2 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the solution to $\min\left\{\|A^3X - A^2B\|_F,\ \mathcal{R}(X) \subseteq \mathcal{R}(A^4)\right\}$ is equal to

$$A^{\textcircled{w}_2}B = \begin{pmatrix} 5 & 3 & 5 & 3 & 2 \\ 4 & 3 & 4 & 5 & 3 \\ 1 & 2 & 2 & 3 & 2 \\ 2 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the solution to $\min\left\{\|A^4X - A^3B\|_F,\ \mathcal{R}(X) \subseteq \mathcal{R}(A^4)\right\}$ is

$$A^{\textcircled{w}_3}B = \begin{pmatrix} 5 & 4 & 5 & 4 & 3 \\ 5 & 3 & 6 & 6 & 3 \\ 1 & 2 & 2 & 3 & 2 \\ 2 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the solution to the constrained matrix equation $A^{m+1}X - A^m B,\ \mathcal{R}(X) \subseteq \mathcal{R}(A^4)$, for each $m \geq \text{ind}(A) = 4$ is

$$A^{\circledR m} B = A^{\mathrm{D}} B = \begin{pmatrix} 6 & 4 & 7 & 5 & 3 \\ 5 & 3 & 6 & 6 & 3 \\ 1 & 2 & 2 & 3 & 2 \\ 2 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 4.2 The test in this example is the real matrix

$$A = \begin{pmatrix} \mathbf{I}_4 & 0.280768 * \mathbf{I}_4 \\ \mathbf{0}_4 & \mathbf{D}_4^1(\{0.45315, 0.67382, 0.417927\}) \end{pmatrix}$$
$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.280768 & 0. & 0. & 0. \\ 0 & 1 & 0 & 0 & 0. & 0.280768 & 0. & 0. \\ 0 & 0 & 1 & 0 & 0. & 0. & 0.280768 & 0. \\ 0 & 0 & 0 & 1 & 0. & 0. & 0. & 0.280768 \\ \hline 0 & 0 & 0 & 0 & 0 & 0.45315 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.67382 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.417927 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Our goal is to perform numerical experiments on real matrices in the floating point arithmetic. The index of A is equal to $\text{ind}(A) = 4$ because of $\{\text{rank}(A), \text{rank}(A^2), \text{rank}(A^3), \text{rank}(A^4), \text{rank}(A^5)\} = \{7, 6, 5, 4, 4\}$.

(a) In the first part of this example we compute the Drazin inverse, the core-EP inverse and m-WGI inverses according to their definitions. The Drazin inverse of A is equal to

$$A^{\mathrm{D}} = A^4 \left(A^9\right)^{\dagger} A^4$$
$$= \left(\begin{array}{cccc} 1. & -3.2266 * 10^{-16} & -1.7000 * 10^{-16} & -6.9389 * 10^{-18} \\ 1.4181 * 10^{-16} & 1. & -9.0206 * 10^{-17} & -2.8103 * 10^{-16} \\ -1.4268 * 10^{-16} & -2.8449 * 10^{-16} & 1. & 4.1633 * 10^{-17} \\ -2.0036 * 10^{-16} & 1.7347 * 10^{-17} & -3.8164 * 10^{-16} & 1. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{array}\right.$$
$$\left.\begin{array}{cccc} 0.280768 & 0.12723 & 0.0857301 & 0.0358289 \\ 3.9817 * 10^{-17} & 0.280768 & 0.189187 & 0.0790663 \\ -4.0060 * 10^{-16} & -9.8030 * 10^{-17} & 0.280768 & 0.11734 \\ -5.6255 * 10^{-17} & -2.0621 * 10^{-17} & -1.2105 * 10^{-16} & 0.280768 \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{array}\right),$$

and the core-EP inverse of A is equal to

$$A^{\textcircled{\dagger}} = A^k\left(A^{k+1}\right)^{\dagger}$$

$$= \begin{pmatrix} 1. & -4.9567*10^{-16} & -2.03396*10^{-16} & 2.8796*10^{-16} & 0. & 0. & 0. & 0. \\ -2.1771*10^{-16} & 1. & -2.4633*10^{-16} & -3.4694*10^{-16} & 0. & 0. & 0. & 0. \\ -1.24033*10^{-16} & -2.6801*10^{-16} & 1. & -1.5959*10^{-16} & 0. & 0. & 0. & 0. \\ -4.3368*10^{-17} & -6.5919*10^{-17} & -2.1858*10^{-16} & 1. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

$$\approx \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}.$$

the WGI (or 1-WGI) inverse of A is given by

$$A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$$

$$= \left(\begin{matrix} 1. & -9.9139*10^{-16} & -4.0679*10^{-16} & 5.7593*10^{-16} \\ -4.35416*10^{-16} & 1. & -4.9266*10^{-16} & -6.9389*10^{-16} \\ -2.4807*10^{-16} & -5.3603*10^{-16} & 1. & -3.1919*10^{-16} \\ -8.6736*10^{-17} & -1.3184*10^{-16} & -4.3715*10^{-16} & 1. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{matrix}\right.$$

$$\left.\begin{matrix} 0.280768 & -2.7835*10^{-16} & -1.1421*10^{-16} & 1.6170*10^{-16} \\ -1.2225*10^{-16} & 0.280768 & -1.3832*10^{-16} & -1.9482*10^{-16} \\ -6.9649*10^{-17} & -1.5050*10^{-16} & 0.280768 & -8.9618*10^{-17} \\ -2.4353*10^{-17} & -3.7016*10^{-17} & -1.2274*10^{-16} & 0.280768 \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{matrix}\right)$$

$$\approx \begin{pmatrix} \mathbf{I}_4 & 0.280768*\mathbf{N}_4^{[1]} \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix}.$$

The 2-WGI inverse (or GG) of A is equal to

$$A^{\circledR_2} = (A^{\textcircled{T}})^3 A^2$$

$$= \left(\begin{array}{cccccccc} 1. & -1.4871*10^{-15} & -6.1019*10^{-16} & 8.6389*10^{-16} & 0.280768 & 0.12723 & -4.5266*10^{-16} & 1.7095*10^{-16} \\ -6.5312*10^{-16} & 1. & -7.3899*10^{-16} & -1.0408*10^{-15} & -1.8338*10^{-16} & 0.280768 & 0.189187 & -3.789*10^{-16} \\ -3.7210*10^{-16} & -8.0404*10^{-16} & 1. & -4.7878*10^{-16} & -1.0447*10^{-16} & -2.7309*10^{-16} & 0.280768 & 0.11734 \\ -1.3010*10^{-16} & -1.9776*10^{-16} & -6.5573*10^{-16} & 1. & -3.6529*10^{-17} & -7.2077*10^{-17} & -2.2152*10^{-16} & 0.280768 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{array} \right)$$

and the 3-WGI inverse of A is

$$A^{\circledR_3} = (A^{\textcircled{T}})^4 A^3$$

$$= \left(\begin{array}{cccccccc} 1. & -1.9828*10^{-15} & -8.1359*10^{-16} & 1.1519*10^{-15} & 0.280768 & 0.12723 & 0.0857301 & 7.11658505897139*10^{-17} \\ -8.7083*10^{-16} & 1. & -9.8532*10^{-16} & -1.3878*10^{-15} & -2.4450*10^{-16} & 0.280768 & 0.189187 & 0.0790663 \\ -4.9613*10^{-16} & -1.0721*10^{-15} & 1. & -6.3838*10^{-16} & -1.3930*10^{-16} & -3.6412*10^{-16} & 0.280768 & 0.11734 \\ -1.7347*10^{-16} & -2.6368*10^{-16} & -8.7430*10^{-16} & 1. & -4.8705*10^{-17} & -9.6103*10^{-17} & -3.1023*10^{-16} & 0.280768 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{array} \right).$$

For each $m \geq \mathrm{ind}(A)$, it is verified that

$$A^{ⓦm} = \left(\begin{array}{cccc} 1. & -2.4785 * 10^{-15} & -1.0170 * 10^{-15} & 1.4398 * 10^{-15} \\ -1.0886 * 10^{-156} & 1. & -1.2317 * 10^{-15} & -1.7347 * 10^{-15} \\ -6.2016 * 10^{-16} & -1.3401 * 10^{-15} & 1. & -7.97973 * 10^{-16} \\ -2.1684 * 10^{-16} & -3.2960 * 10^{-16} & -1.0929 * 10^{-15} & 1. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{array}\right.$$

$$\left.\begin{array}{cccc} 0.280768 & 0.12723 & 0.0857301 & 0.0358289 \\ -3.0563 * 10^{-16} & 0.280768 & 0.189187 & 0.0790663 \\ -1.7412 * 10^{-16} & -4.5515 * 10^{-16} & 0.280768 & 0.11734 \\ -6.0882 * 10^{-17} & -1.20123 * 10^{-16} & -3.8779 * 10^{-16} & 0.280768 \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{array}\right).$$

The Frobenius norm of $A^{ⓦm} - A^{\mathrm{D}}$ for $m \geq \mathrm{ind}(A)$ is equal to $\|A^{ⓦm} - A^{\mathrm{D}}\|_F = 3.79579 * 10^{-15}$.

(b) In the second part we compute the class of m-WGI inverses according to Corollary 2.1 and Corollary 2.2. The result of `{Q,R,P}=QRDecomposition[MatrixPower[A,4]]` is the ordered triple

$$\{Q, R, P\} = \left\{ I_8, \begin{pmatrix} 1. & 0. & 0. & 0. & 0.280768 & 0.12723 & 0.0857301 & 0.0358289 \\ 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.189187 & 0.0790663 \\ 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.11734 \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}, I_8 \right\}.$$

The skinny parts of Q and R are equal to

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1. & 0. & 0. & 0. & 0.280768 & 0.12723 & 0.0857301 & 0.0358289 \\ 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.189187 & 0.0790663 \\ 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.11734 \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 \end{pmatrix}.$$

According to Corollary 2.2, our further interest is the matrix

$$Q_1^*(Q_1 A^4 Q_1^*)^{-1} Q_1 A^3 = A^{(2)}_{\mathcal{R}(Q_1^*),\mathcal{N}(Q_1 A^3)}$$
$$= \begin{pmatrix} 1. & 0. & 0. & 0. & 0.280768 & 0.12723 & 0.0857301 & 0. \\ 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.189187 & 0.0790663 \\ 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.11734 \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

which approximates $A^{\circledR 3}$, since $\|A^{\circledR 3} - Q_1^*(Q_1A^4Q_1^*)^{-1}Q_1A^3\|_F = 3.59685 * 10^{-15}$. It is a confirmation of Corollary 2.2.

The next verification of the statement in Corollary 2.2 is the matrix

$$Q_1^*(Q_1A^3Q_1^*)^{-1}Q_1A^2 = A^{(2)}_{\mathcal{R}(Q_1^*),\mathcal{N}(Q_1A^2)} = \begin{pmatrix} 1. & 0. & 0. & 0. & 0.280768 & 0.12723 & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.189187 & 0. \\ 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 & 0.11734 \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

which satisfies $\|A^{\circledR 2} - Q_1^*(Q_1A^3Q_1^*)^{-1}Q_1A^2\|_F = 2.6825 * 10^{-15}$. Finally,

$$Q_1^*(Q_1A^2Q_1^*)^{-1}Q_1A = A^{(2)}_{\mathcal{R}(Q_1^*),\mathcal{N}(Q_1A)} = \begin{pmatrix} 1. & 0. & 0. & 0. & 0.280768 & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0.280768 & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.280768 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

approximates $A^{\circledR}$, since $\|A^{\circledR} - Q_1^*(Q_1A^2Q_1^*)^{-1}Q_1A\|_F = 1.73739 * 10^{-15}$.

Example 4.3 Our intention in this example is to perform numerical experiments on the matrix

$$A = \begin{pmatrix} \mathbf{I}_3 & \iota * \mathbf{I}_3 \\ \mathbf{0}_3 & (1+\iota) * \mathbf{D}_3^1 \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \iota & 0 & 0 \\ 0 & 1 & 0 & 0 & \iota & 0 \\ 0 & 0 & 1 & 0 & 0 & \iota \\ \hline 0 & 0 & 0 & 0 & 1+\iota & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\iota \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

with complex entries involving the imaginary unit ι. Rank identities {rank(A), rank(A^2), rank(A^3), rank$(A$
{5, 4, 3, 3} confirm $k = \mathrm{ind}(A) = 3$.

(a) Exact calculation gives

$$A^{\mathrm{D}} = A^3 \left(A^7\right)^{\dagger} A^3 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \iota & -1+\iota & -2 \\ 0 & 1 & 0 & 0 & \iota & -1+\iota \\ 0 & 0 & 1 & 0 & 0 & \iota \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$A^{\textcircled{\dagger}} = A^3 \left(A^4\right)^{\dagger} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix},$$

$$A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \iota & 0 & 0 \\ 0 & 1 & 0 & 0 & \iota & 0 \\ 0 & 0 & 1 & 0 & 0 & \iota \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} \mathbf{I}_3 & \iota * \mathbf{N}_3^{[1]} \\ \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix},$$

$$A^{\textcircled{w}_2} = (A^{\textcircled{\dagger}})^3 A^2 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \iota & -1+\iota & 0 \\ 0 & 1 & 0 & 0 & \iota & -1+\iota \\ 0 & 0 & 1 & 0 & 0 & \iota \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$A^{\textcircled{w}_m} = (A^{\textcircled{\dagger}})^{m+1} A^m = A^{\mathrm{D}}, \quad m \geq 3 = \mathrm{ind}(A).$$

(b) The output of *Mathematica* command of `{Q,R,P}=QRDecomposition[MatrixPower[A,3]]` is the ordered triple

$$\{Q, R, P\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & i & -1+i & -2 \\ 0 & 1 & 0 & 0 & i & -1+i \\ 0 & 0 & 1 & 0 & 0 & i \end{pmatrix}, I_6 \right\}$$

which defines the QR decomposition (2.5). Further calculation gives

$$Q^*(QA^3Q^*)^{-1}QA^2 = A^{(2)}_{\mathcal{R}(Q^*),\mathcal{N}(QA^2)} = A^{\textcircled{w}_2}$$

and

$$Q^*(QA^2Q^*)^{-1}QA = A^{(2)}_{\mathcal{R}(Q^*),\mathcal{N}(QA)} = A^{\textcircled{w}},$$

which is consistent with Corollary 2.2.

5 Conclusion

The *m*-weak group inverse (*m*-WGI) was introduced in [34] as an extension of the concept of the weak group inverse (WGI). Some properties of the *m*-WGI were presented in [9, 15, 26, 34]. In this paper, we introduce several additional representations for the m-weak group inverse in terms of full rank factorizations of rank-invariant matrix powers A^k, $k \geq \text{ind}(A)$. An effective computational procedure for computing m-weak group inverse in terms of the QR decomposition of the matrix A^k, $k = \text{ind}(A)$ is presented. Our second motivation is the constrained matrix minimization problem considered in [26] and solved in terms of the WGI. Applying the m-weak group inverse, we present the uniquely determined solution to the restricted optimization problem in the Frobenius norm: min $\|A^{m+1}X - A^m B\|_F$ provided that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, where $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$ and $\text{ind}(A) = k$. Some well-known results related to the WGI and the Drazin inverse are special cases of the considered minimization problem.

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Data availability The data that support the findings of this study are available on request from the corresponding author.

Declarations

Conflict of interest The authors declare no conflict of interest.

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