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P. STANIMIROVIĆ
D. MOSIĆ
A. STUPINA
R. BEHERA

SOLVING MATRIX APPROXIMATION PROBLEMS USING GENERALIZED INVERSES

MONOGRAPH

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Reviewers:

Kazakovtsev L.A., Doctor of Technical Science, Professor, Professor of the Department of Digital Control Technologies (Siberian Federal University);

Masich I.S., Doctor of Technical Science, Associate Professor, Professor of the Department of System Analysis (Reshetnev Siberian State University of Science and Technology)

Stanimirović P.

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Our main intention in this monograph is to solve the problems of matrix approximation and optimization without restrictions and with restrictions using generalized inversion operations as a tool of linear algebra.

The inverse matrix is defined only for square non-singular matrices. On the other hand, generalized inversions are very different. The broadest definition of a generalized inverse matrix states that this matrix:

– exists for a larger class of rectangular and (or) singular matrices;
– has some properties of the ordinary inverse;
– for a given square nonsingular matrix reduces to the ordinary inverse.

The Moore-Penrose inverse (or pseudoinverse) is useful in all kinds of least squares problems. The Drazin inverse is applicable mainly in population modeling, Markov chains, and singular systems of linear differential equations.

This monograph is aimed to mathematics, engineering graduate students and researchers in the areas of numerical linear algebra, optimization, dynamical systems, control systems, signal processing.

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Министерство науки и высшего образования Российской Федерации
Сибирский федеральный университет

П. СТАНИМИРОВИЧ
Д. МОСИЧ
А. СТУПИНА
Р. БЕХЕРА

РЕШЕНИЕ ЗАДАЧ МАТРИЧНОГО ПРИБЛИЖЕНИЯ С ИСПОЛЬЗОВАНИЕМ ОБОБЩЕННЫХ ОБРАТНЫХ

МОНОГРАФИЯ

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Рецензенты:

Казаковцев Л.А., доктор технических наук, профессор, профессор кафедры цифровых технологий управления Сибирского федерального университета;

Масич И.С., доктор технических наук, доцент, профессор кафедры системного анализа Сибирского государственного университета науки и технологий имени академика М.Ф. Решетнева

Станимирович П.

С76 Решение задач матричного приближения с использованием обобщенных обратных : монография / П. Станимирович, Д. Мосич, А. Ступина, Р. Бехера. — Москва : ИНФРА-М, 2023. — 165 с. — (Научная мысль).

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При написании монографии основной целью авторов было решение задач матричной аппроксимации и оптимизации без ограничений и с ограничениями с использованием обобщенных операций обращения в качестве инструмента линейной алгебры.

Обратная матрица определяется только для квадратных не сингулярных матриц. В то же время обобщенные инверсии сильно отличаются. Самое широкое определение обобщенной обратной матрицы утверждает, что это матрица, которая:

– существует для большего класса прямоугольных и (или) сингулярных матриц;

– обладает некоторыми свойствами обычного обратного преобразования;

– для данной квадратной не сингулярной матрицы уменьшается до обычной обратной.

Инверсия Мура — Пенроуза (или псевдоинверсия) полезна во всех видах задач наименьших квадратов. Обратная матрица Дразина применима главным образом в популяционном моделировании, марковских цепях и сингулярных системах линейных дифференциальных уравнений.

Ориентирована на аспирантов и исследователей в области математики и инженерии, специализирующихся на методах числовой линейной алгебры, оптимизации, динамических систем, систем управления, обработки сигналов.

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Contents

Preface

Linear algebra (Latin: linealis, belonging to a line) is a mathematical discipline that deals with vectors and matrices, and in particular with vector space and linear transformations. Linear algebra is the center of mathematics and applied mathematics. Abstract algebra is created by omitting the axioms of the vector space. Functional analysis studies an infinite no-dimensional version of the theory of vector spaces. Unlike other parts of mathematics, where new and never-before-seen problems often appear, in linear algebra this is not a frequent occurrence. Its value lies in its applicability, ranging from engineering, analytical geometry, mathematical physics, abstract algebra, as well as applications in economics, programming and computing. The use of matrices in quantum mechanics, special relativity, and statistics has helped expand the subject of linear algebra beyond pure mathematics. The development of computers has led to more research into efficient algorithms for Gaussian elimination and matrix decomposition, and linear algebra has become an essential tool for modeling and simulations. In image reconstruction, each image is viewed as a matrix and image blur is modeled with specific matrix equations. Graph theory can hardly be imagined without matrices and linear algebra.

The concept of generalized inverse matrix is an extension of the concept of the inverse matrix applicable even to square singular matrices and rectangular matrices. Many definitions of generalized inverses are presented so far, and all of which reduce to the usual inverse when the matrix is square and nonsingular. Generalized inverses are included an extensive variety of mathematical fields, for example, matrix theory and operator theory. Generalized inverses are very powerful tools in many branches of mathematics, technics and engineering. The most frequent application of generalized inverses is in finding solution of many matrix equations and systems of linear equations. There are a lot of other mathematical and technical disciplines in which generalized inverses play an important role. Some of them are: estimation theory (regression), computing polar decomposition, electrical circuits (networks) theory, automatic control theory, filtering, difference equations, robotics, pattern recognition, image restoration. Since 1955, thousands of papers have been published discussing various theoretical and computational aspects of generalized inverses and their applications.

The Moore-Penrose inverse (or pseudoinverse) is useful in all kinds of least squares problems. The Drazin inverse is applicable mainly in population modelling, Markov chains, and singular systems of linear differential equations.

The broadest definition for a generalized inverse of a matrix says that it is a matrix which:
- exists for a larger class of matrices than the ordinary inverse does (for example, for rectangular and/or singular matrices);
- has some properties of the ordinary inverse;
- for a given square nonsingular matrix it reduces to the ordinary inverse.

Different types of generalized inverses have been introduced with the purpose of defining a generalized inverse which will have as many properties as the ordinary matrix inverse. The first such inverse, named by the scientists who worked on this topic, is the Moore-Penrose inverse, or pseudoinverse. Nowadays, the theory of generalized inverses recognizes many types of generalized inverses different than the Moore-Penrose inverse. Main of them are: the Drazin inverse, the group inverse, the weighted Moore-Penrose inverse, $\{i, j, k\}$-inverses, the Bott-Duffin inverse etc.

Also, a number of monographs has been written [10, 27, 8, 12, 160, 145, 163]. Ben Israel in his survey paper [7] observed about 2000 articles and 15 books on generalized inverses of matrices and linear operators. Now, it is difficult to give even an approximate number of articles devoted to generalized inverses. It is justifiably to say that the theory of generalized inverses extensively grows and becomes an important part of mathematics as well as important part of many applicable scientific areas, such as computer science, electrical engineering, etc. For more information of he history of generalized inverses the reader is referred to two survey papers [7] and [6]. Global overview of various applications of generalized inverses can be found in [7].

Real interest in the study generalized inverses appeared after the publications of Penrose's paper [122]. In the word of Nashed, that paper represents the renaissance in the development of theory of generalized inverses and it may be said that as a mathematical area generalized inversion was inaugurated in 1955 by Penrose [122]. The generalized inversion is listed in the 2010 Mathematics Subject Classification under 15A09: Matrix inversion, generalized inverses and 65F20: overdetermined systems, pseudoinverses.

Since the publication of [122], generalized inverse become a permanent, distinguished an active area in mathematics. Many types of generalized inverses have various applications such as linear estimation, differential and difference equations, Markov chains, graphics, cryptography, coding theory, robotics, incomplete data recovery, sociology, demography and many other fields. A special case of the Drazin inverse, called Group inverse, has found application in characterizing the sensitivity of the stationary probabilities to perturbations in the underlying transition probabilities. Finally, the group inverse has recently proven to be fundamental in the analysis of Google's PageRank search engine. Generalized inverses play an important role in finding solutions of many stochastic models, in particular Markov chains in discrete or continuous time and Markov renewal processes [52]. The Drazin inverse has been successfully and extensively applied in different fields of science; for example, in finding closed form solutions of singular differential equations with matrix coefficients, in solving difference equations, in Markov chains, multibody system dynamics as well as in finding solutions of various iterative methods. The Moore-Penrose inverse has found a wide range of applications in many areas of science and became a useful in finding least squares solutions of linear systems, in optimization problems, in data analysis, in finding the solution of linear integral equations, etc. Global overview of various applications of generalized inverses can be found in [7].

Usually, an optimization method is an iterative method for finding the minimum or maximum of some optimization problem. Namely, given an initial point x_0, an iterative sequence x_k is generated by a given iterative rule, such that the sequence x_k converges to the optimal solution of the problem. A typical behavior of an algorithm which is regarded as acceptable is that the iterates x_k move steadily towards the neighborhood of a local optimizer x, and then rapidly converge to the point x.

The optimization theory represents a very important mathematical discipline and finds great application, not only in the theory of applied mathematics, but also in many practical disciplines such as: production, aviation, management, sociology, genetic etc. Moreover, the process of evolution reveals that follows optimization.

Nonlinear optimization and approximation theory are two scientific fields related to generalized inverses. It is known that the calculation of the generalized inverse matrix is often involved in finding solutions for some optimization and approximation models. On the other hand, the calculation of the inverse and pseudoinverse matrix can be defined based on certain optimization models. Although the optimization theory is a part of everyday life for a very long time, this science has faced an important development in the last five decades. The subject is involved in the process of finding optimal solution of problems which are defined mathematically, i.e., given a practical problem, the "best" solution to the problem can be found from lots of schemes by means of scientific methods and tools. It involves the study of optimality conditions of the problems, the construction of model problems, the determination of algorithmic method of solution, the establishment of convergence theory of the algorithms, and numerical experiments with typical problems and real life problems.

The term "good algorithm" assumes the following properties:

- Robustness, since it should perform well on a wide variety of problems in their class, for all reasonable choices of the initial variables.
- Efficiency, since it should not require too much computer time or storage.
- Accuracy, since it should be able to identify a solution with precision, without being overly sensitive to errors in the data, or to the arithmetic rounding errors that occur when the algorithm is implemented on a computer.

Nonlinear optimization and approximation theory are two scientific areas related to generalized inverses. All these goals are usually conflict, so, tradeoffs between the different types of good properties is a central issue in numerical optimization.

This monograph is aimed to mathematics and engineering graduate students and researchers in the areas of numerical linear algebra, optimization, dynamical systems, control systems, signal processing. It can also be used as a text or reference for many graduate courses or as a reference for many courses in postgraduate levels in computer science, mathematics or in technical faculties. The reader should be familiar with basic linear algebra, matrix theory, mathematical and functional analysis. Knowledge in *Mahematica* programming package is desirable. We believe that the book should be of use for many researchers, students in applied mathematics, statistics, engineering, and many other scientific disciplines.

The global organization of the monograph's chapters and sections is as follows.

Chapter 1 introduces basic notions, notations and results from the matrix theory, generalized inverses and nonlinear optimization. This chapter starts from basic notions in matrix theory. Introduction to generalized inverses and basic properties of main generalized inverses are analysed in Sections 1.2 and 1.3. Main facts on idempotent matrices and projectors are restated in Section 1.4. Least squares and minimal norm properties of the Moore-Penrose inverse are surveyed in Section 1.5. Minimal properties of Drazin inverse and outer inverses are surveyed in Section 1.6. Last section in this chapter shortly restates relationships between optimization theory and generalized inverses.

A brief overview of *composite generalized inverses* is presented in the second chapter. Involved sections restate main properties, representations and characterizations of proper combinations of the Moore-Penrose inverse and outer inverses with prescribed range and null space. First section is aimed to survey of composite outer inverses involving the Moore-Penrose inverse. This class involves very important generalized inverses, such as the core and core-EP invese, DMP and MPD inverse. Then Section 2.2 presents overview of remaining composed generalized inverses, such as the CMP inverse, dual core-EP inverse, weak group inverse, the *-CEPMP, W-weighted Drazin inverse, and W-weighted core-EP inverse. Subsequent sections describe OMP, MPO and MPOMP inverses as proper combinations of outer generalized inverses with the Moore-Penrose inverse.

Chapter 3 is aimed to *least squares* properties of generalized inverses. Section 3.1 investigates least squares solutions and best approximate solutions of main generalized inverses. Main part of this research are minimal properties of the Moore-Penrose inverse. Section 3.2 investigates analogous properties of the Drazin inverse. Least-square properties of outer inverses are considered in Section 3.3. Finally, least-square properties of composition of outer inverses with main generalized inverses (the Moore-Penrose and the Drazin inverse) are considered in Section 3.4. This section shows that each kind of generalized inverses is related to appropriate matrix equation and/or linear system.

Solvability of approximation problems]Solvability of matrix approximation problems is the topic of Chapter 4. Solvability of approximation problems based on core-EP inverse are considered in section 4.1 and 4.2. Core-EP inverse solution with least-squares solutions is considered in Section 4.3.

Various generalizations of composite inverses are presented in Chapter 5. Characterizations of g-core-EP inverse are given in Section 5.1. Precisely, an extension of the core-EP inverse (termed as the g-core-EP inverse) for a rectangular matrix in terms of the Moore-Penrose inverse of a corresponding matrix and the outer inverse. Applications of g-core-EP inversesApplications of g-core-EP and *g-core-EP inverses are given in Section 5.3. Generalizations of OMP, OMP and MPOMP inverses are defined in Section 5.5. Section 5.7 investigates extensions of the generalized CEP inverse, while extensions of dual generalized CEP inverses are considered in Section 5.8. Algorithms and examples of presented generalizations are involved in Section 5.9. Some applications of the proposed Φ-GCEP and Φ-*GCEP inverses in finding solutions to several linear vector equations are presented in Section 5.10.

Outer-star and star-outer matrices are objective of Chapter 6. Particularly, Section 6.1 investigates characterizations of outer-star matrices, and Section 6.2 considers representations of outer-star matrices. Further, representations of Drazin-star matrices are presented in Section 6.3 and roup-star and star-group matrices in Section 6.4. Applications of outer-star and star-outer matrices are given in Section 6.5.

Minimal rank properties of outer inverses are considered in Chapter 7. Subsequent sections consider minimal rank outer inverses with prescribed range, prescribed null space and both prescribed range and null space.

Chapter 1

Introduction

Main intention in this monograph is solving unconstrained and constrained matrix approximation and optimization problems using generalized inverses. The investigation of outer generalized inverses which are defined as proper composition of the Moore-Penrose inverse and some particular outer inverses, such as the Drazin inverse or the group inverse has attracted a great popularity in last years.

Computation of the Moore–Penrose inverse of a matrix is mainly done via the calculation of either a full-rank decomposition or the singular value decomposition (SVD). Further, the SVD is based on singular values. On the other hand, representations and computation of the Drazin inverse are related to the matrix index and the Jordan canonical form. In addition, predefined range and/or null space of outer generalized inverses is important requirement in their definition. Finally, inner inverses (and particularly the Moore-Penrose inverse) are usable in solving systems of linear equations and matrix equations. Because of that, it is important to introduce these notions before detailed introduction of various generalized inverses.

1.1 Basic notions from matrix theory

It is necessary to mention several common and usual notations. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers respectively. As usual, $\mathbb{C}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$) and $\mathbb{C}_r^{m\times n}$ (resp.) denote the set of all complex (resp. real) $m \times n$ matrices and all complex (real) $m \times n$ matrices of rank r, respectively. It is well known that a matrix $A \in \mathbb{C}^{m\times n}$ represents a matrix form of a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ with respect to the standard basis of $\mathbb{C}^n$ and $\mathbb{C}^m$. For a given matrix A, by A^*, $\mathcal{R}(A)$, $\mathrm{rank}(A)$ and $\mathcal{N}(A)$ we denote the conjugate transpose, the range, the rank and the null space of $A \in \mathbb{C}^{m\times n}$. Denote by $\mathbb{F}$ the arbitrary field. Although only relevant fields in this book are $\mathbb{R}$ and $\mathbb{C}$, all statements containing $\mathbb{F}$ are valid for an arbitrary field. In the similar manner, $\mathbb{F}^{m\times n}$ denotes set of $m \times n$ matrices over the field $\mathbb{F}$. Identity matrix of the format $n \times n$ will be denoted by I_n (or simply I when dimensions are known) where diagonal matrix whose diagonal entries are $d_1, d_2, \dots, d_n$ will be denoted by $\mathrm{diag}(d_1, d_2, \dots, d_n)$. An appropriate zero matrix will be denoted by O (or simply by 0). Also, $|A|$ denotes the determinant of A.

Definition 1.1.1. (Definition of inner product) *Let V be a complex vector space. An* inner product *$\langle x, y\rangle$ is a function $V \times V \to \mathbb{C}$ which satisfies the following properties:*
(a) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ *(linearity);*
(b) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ *(Hermitian symmetry);*
(c) $\langle x, x\rangle \geq 0,\ \langle x, x\rangle$ *if and only if* $x = 0$ *(positivity);*
for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$.

The standard inner product in $\mathbb{C}^n$ is defined by

$$\langle x, y\rangle = y^* x = \sum_{i=1}^{n} x_i \overline{y_i},$$

for arbitrary vectors $x = (x_i), y = (y_i) \in \mathbb{C}^n$. If $\langle x, y\rangle$ is an inner product on $\mathbb{C}^n$, then

$$\|x\| := \sqrt{\langle x, x\rangle}$$

is a norm on $\mathbb{C}^n$. This vector norm possesses the following properties:
(i) $\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b\rangle$,
(ii) $\|a\| = 0 \Longleftrightarrow a = 0$.

The distance between two vectors can be defined by $d(a,b) = \|a-b\|$. Main properties of the distance d are as follows:
(i) $d(a,b) \geq 0$, $d(a,b) = 0 \Longleftrightarrow a = b$,
(ii) $d(a,b) = d(b,a)$,
(iii) $d(a,b) + d(b,c) \geq d(a,c)$.
The last three properties above are called the metric (or distance) axioms.

The matrix product (when it exists) satisfies the following:

1° $(AB)C = A(BC)$;

2° $A(B+C) = AB + AC$, $(B+C)A = BA + CA$;

3° $\alpha(AB) = (\alpha A)B = A(\alpha B)$, gde je $\alpha \in F$ skalar;

4° $IA = AI = A$, where I jdenotes the identity matrix of appropriate dimensions.

A matrix norm of $A \in \mathbb{C}^{m\times n}$ is denoted by $\|A\|$ and defined as a function $\mathbb{C}^{m\times n} \to \mathbb{R}$ satisfying
$\|A\| \geq 0$, $\|A\| = 0$ only if $A = O$,
$\|\alpha A\| = |\alpha|\|A\|$,
$\|A+B\| \leq \|A\| + \|B\|$,
for all $A, B \in \mathbb{C}^{m\times n}$, $\alpha \in \mathbb{C}$.
If, in addition, $\|AB\| \leq \|A\|\|B\|$, then $\|\|$ is a multiplicative norm.

Next we give definitions for the notions which are usually related to a given matrix and which are frequently used further in the text.

Definition 1.1.2. *A square matrix $A \in \mathbb{C}^{n\times n}$ ($A \in \mathbb{R}^{n\times n}$) is*

(a) *Hermitian (self-adjoint) if $A^* = A$ ($A^{\mathrm{T}} = A$),*

(b) *normal, if $A^*A = AA^*$ ($A^{\mathrm{T}}A = AA^{\mathrm{T}}$),*

(c) *lower-triangular, if $a_{ij} = 0$ for $i > j$,*

(d) *upper-triangular, if $a_{ij} = 0$ for $i < j$,*

(e) *positive semi-definite, if $\Re(x^*Ax) \geq 0$ for all $x \in \mathbb{C}^n$,*

(f) *positive definite, if $\Re(x^*Ax) > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$.*

The notion $\Re(z)$ means the real part of a complex number z.

By $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$ is denoted the trace of a square matrix $A \in \mathbb{C}^{n\times n}$.

Let E^n denote the set of all n-component vectors with entries over E. Assume that there exists r linearly independent vectors in a subspace W of E^n, while any set of $r+1$ vectors is linearly dependent. Then the dimensionality of W is said to be r, and it is denoted by $\dim(W) = r$. The maximal number of linearly independent columns of a matrix A is called *rank* of A and it is denoted by $\mathrm{rank}(A)$.

Definition 1.1.3. *Let $A\mathbb{C}^{m\times n}$ be the set of complex matrices of type $m\times n$ over complex numbers $\mathbb{C}$ and $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ the set of natural numbers. Rank of a matrix A is the mapping*

$$\mathrm{rank} : \mathbb{C}^{m\times n} \to \mathbb{N}_+$$

defined by

(i) $\mathrm{rank}(\mathbf{0}) = 0$, *where $\mathbf{0}$ is the zero (null) matrix;*

(ii) $\mathrm{rank}(A) = r$, *if there is a minor of order r of matrix A that is different from zero, and all minors of order k, where $r+1 \leq k \leq \min\{m,n\}$, if they exist, are equal to zero.*

Based on Definition 1.1.3 of matrix rank, for the matrix A of type $m \times n$, it immediately follows that

$$0 \leq \mathrm{rank}(A) \leq \min\{m,n\}.$$

Definition 1.1.4. *Each minor $M_r \neq 0$ of order r of a given matrix A for which* $\mathrm{rank}(A) = r$ *is called a basis or basis minor of that matrix. The types and columns of the matrix A in whose intersections there are elements of the basic minor M_r are called the basic types and basic columns of the matrix A (ie, basic types and basic columns).*

The importance of basic rows and basic columns in relation to other rows and columns of a given matrix is described by the following result on the basic minor.

For a given matrix A važi:
(i) basic rows (basic columns) of the matrix A are linearly independent;
(ii) rows (columns) of the matrix A that are not basic, if any, are a linear combination of the basic rows (basic columns).

Definition 1.1.5. *Let $A \in \mathbb{C}^{m\times n}$. A real or complex scalar λ which satisfies the following equation*

$$Ax = \lambda x, \qquad i.e., \qquad (A - \lambda I)x = 0,$$

is the eigenvalue of A, and x is the eigenvector of A corresponding to λ.

Definition 1.1.6. *For any matrix $A \in \mathbb{C}^{m\times n}$, the null space $\mathcal{N}(A)$ is defined as the inverse image of the zero vector 0, i.e.*

$$\mathcal{N}(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}.$$

Also define the range $\mathcal{R}(A)$ as the set of all images

$$\mathcal{R}(A) = \{y \in \mathbb{C}^m \mid y = Ax \text{ for some } x \in \mathbb{C}^n\}.$$

The dimension of range $\mathcal{R}(A)$ is called *rank* of the matrix A and denoted by $\operatorname{rank}(A)$. The column rank of A is the dimension of the column space of A, while the row rank of A is the dimension of the row space of A. The column rank and the row rank are always equal.

Proposition 1.1.1. *Let $A \in \mathbb{C}^{m\times p}, B \in \mathbb{C}^{m\times q}$. The matrix rank possesses the following properties:*
(i) *If U and V are nonsingular, then* $\operatorname{rank}(UAV) = \operatorname{rank}(A)$.
(ii) $\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$.

Matrix index is another important characteristics of matrices.

Proposition 1.1.2. *For every $A \in \mathbb{C}^{n\times n}$ there exists an integer k such that* $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$.

Definition 1.1.7. *Let $A \in \mathbb{C}^{n\times n}$. Smallest integer k such that holds* $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$ *is called the index of A and it is denoted by* $\operatorname{ind}(A) = k$.

Note that $\operatorname{ind}(A) = 0$ if A is regular and otherwise $\operatorname{ind}(A) \ge 1$. The notion of the matrix index plays an important role in studying generalized inverses.

The eigenvalues and eigenvectors of a matrix are crucial notions in matrix theory. For example, they represent a tool which enables to inquire the structure and main characteristics of a matrix. For example, the eigenvalues can be used as a test for regularity of a matrix. In addition, if a given square matrix of complex numbers is self-adjoint, then there exist a basis of $\mathbb{C}^m$ and a basis of $\mathbb{C}^n$, consisting of distinct eigenvectors of A, with respect to which the matrix A can be represented as a diagonal matrix. Since not every matrix has enough distinct eigenvectors to enable its nice decomposition, the following generalization of the eigenvalue is useful in order to resolve this problem.

Definition 1.1.8. *Let $A \in \mathbb{C}^{n\times n}$ and λ is an eigenvalue of A. A vector x is called generalized eigenvector of A of a grade p corresponding to λ, or λ-vector of A of a grade p if it satisfies the equation*

$$(A - \lambda I)^p x = 0.$$

Proposition 1.1.3. (Full rank factorization)
(*a*) *Each $m \times n$ matrix A of rank r can be written in the form $A = PQ$, where $P \in \mathbb{C}_r^{m\times r}$ and $Q \in \mathbb{C}_r^{r\times n}$ are matrices of rank r.*
(*b*) *The matrix A can be represented as a sum of r matrices of rank* 1.

Proposition 1.1.4. (LU and Cholesky factorization)
For every square square matrix $A \in \mathbb{C}^{n\times n}$ there exists a lower triangular matrix L and an upper triangular matrix U such that $l_{ii} = 1$ for all $i = 1, 2, \ldots, n$. This factorization is known as the LU factorization. Moreover, if A is Hermitian and positive definite, then $U = L^*$, and the unduced factorization $A = LL^*$ is called the Cholesky factorization.

The LU decomposition of the square matrix A consists in its representation as a product of two matrices $A = LU$, where the matrix L is lower-triangular with units on the main diagonal, and the matrix U is upper-triangular. If such a decomposition is known, it can be used to solve the system of linear equations $Ax = b$. Namely, then it is valid

$$Ax = (LU)x = L(Ux) = b.$$

From here it can be seen that by solving the system

$$Ly = b$$

and then the system

$$Ux = y$$

it is possible to get a solution to the starting system.

Relying on the LU decomposition, it will be possible to calculate it and the inverse of the matrix. Let $e_1, \dots, e_n$ columns be the only čne matrices I_n of dimension $n \times n$. Then the ith column of a^i inverse A^{-1} is obtained by solving the system

$$Ly = e_i$$

and then the system

$$Ua^i = y.$$

For each matrix, there is a basis composed of generalized eigenvectors in relation to which the matrix can be represented in the Jordan form. The Jordan decomposition is stated in the following statement.

Proposition 1.1.5. (Žordanova dekompozicija) *Let $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ be different eigenvalues of the square matrix $A \in \mathbb{C}^{n\times n}$. Then A is similar to the diagonal matrix J with Jordan blocks on its diagonal, i.e., there exists a nonsingular matrix P such that*

$$A = PJP^{-1} = P \begin{bmatrix} J_{k_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{k_2}(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k(\lambda_p) \end{bmatrix} P^{-1},$$

where the Jordan bloks $J_{k_i}(\lambda_i)$ are defined as

$$J_{k_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}.$$

The Jordan matrix J is defined uniquely up to the rearrangement of its diagonal blocks.

Note that the characteristic polynomial $k \times k$ of the Jordan block is equal to $(x - \lambda_i)^{k_i}$. This means that each Jordan block $J_{k_i}(\lambda_i)$ has its own value λ_i.

Let $A \in \mathbb{C}^{m\times n}$ be the given matrix. Then, let the vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^n$ and a real number $\sigma \geq 0$ be such that

$$Av = \sigma u, \quad A^*u = \sigma v. \tag{1.1}$$

Then σ is called the singular value of A. In addition, the vectors u and v are called the left and right singular vectors of the matrix A, respectively.

Based on (1.1), it follows

$$A^*Av = \sigma^2 v, \quad AA^*u = \sigma^2 u,$$

which implies that the real number σ^2 is an eigenvalue of A^*A and AA^*.

Proposition 1.1.6. *Let $A \in \mathbb{C}^{m\times n}$ be a matrix of rank r and let $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$ be the nonzero eigenvalues of the matrix AA^* (ie A^*A). There are exactly $r = \operatorname{rank}(A)$ nonzero singular values of the matrix A, which are denoted by $\sigma_i(A)$, $i = 1, \dots, r$, and which are equal to*

$$\sigma_i(A) = \sqrt{\lambda_i}, \quad i = 1, \dots, r.$$

In short, the singular values of the matrix A are defined by

$$\sigma_i(A) = \lambda_i(AA^*) = \lambda_i(A^*A).$$

Singular values fulfill basic properties

$$\sigma_i(A) = \sigma_i(A^{\mathrm{T}}) = \sigma_i(A^*) = \sigma_i(\overline{A}).$$

For all unitary matrices $U \in \mathbb{C}^{m\times m}$ and $U \in \mathbb{C}^{n\times n}$ it is fulfilled

$$\sigma_i(A) = \sigma_i(UAV).$$

Proposition 1.1.7. (Singular Value Decomposition) *Let $A \in \mathbb{C}_r^{m\times n}$ be an arbitrary $m \times n$ matrix of rank r. Then there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that A can be represented in the form*

$$A = U \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} V^*, \tag{1.2}$$

where Σ is dilagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_\mathrm{r}), \ \ \sigma_\mathrm{i} = \sqrt{\lambda_\mathrm{i}}$$

and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ are nonzero eigenvalues of A^*A.

Let us denote by U_r the submatrix consisting of the first r columns of the matrix U and by V_r submatrix consisting of the first r types of the matrix V. Then (1.2) can be written in a more efficient form

$$A = U_r \Sigma V_r.$$

Proposition 1.1.8. *Let the matrix $A \in \mathbb{C}_r^{m\times n}$ of rank r be given. Then there exist unitary matrices $Q \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{C}^{n\times n}$ such that $A = Q^*RP^*$ such that R is the matrix of the form*

$$R = \begin{bmatrix} R_{11} & O \\ O & O \end{bmatrix} = \begin{bmatrix} R_1 \\ O \end{bmatrix} \in \mathbb{C}^{m\times n}, \quad R_{11} \in \mathbb{C}_k^{k\times k},$$

and O denotes corresponding zero block.

In the case where A is of full column rank ($r = n \leq m$), the following statement about QR factorization holds.

Proposition 1.1.9. *Let the matrix $A \in \mathbb{C}_r^{m\times r}$ of rank r be given. Then there exist unitary matrices $Q \in \mathbb{C}^{m\times m}$ so that holds $A = Q^*R$ where R is the shape matrix*

$$R = \begin{bmatrix} R_1 \\ O \end{bmatrix},$$

where $R_1 \in \mathbb{C}^{n\times n}$ is upper-trapezoidal matrix of the form

$$R_1 = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{12} & \cdots & r_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

1.2 Introduction to generalized inverses

For the beginning, we restate facts related to the inverse of a given square matrix.

Definition 1.2.1. *The inverse of a given matrix square regular matrix $A \in \mathbb{C}^{n\times n}$ A is a square matrix A^{-1} satisfying the following equalities:*

$$AA^{-1} = I, \quad A^{-1}A = I.$$

Proposition 1.2.1. *A square matrix $A \in \mathbb{C}^{n\times n}$ has a unique inverse if and only if* $\det(A) \neq 0$, *in which case A is nonsingular (regular) matrix.*

In order to distinguish between generalized inverses, the inverse of a matrix defined with Definition 1.2.1 will be called the ordinary (or usual) inverse.

As previously mentioned, the main idea of defining generalized inverses originates from the need to solve the problem of finding a solution of the following linear system

$$Ax = b, \tag{1.3}$$

where $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. In the case when the matrix A from the system (1.3) is nonsingular, the vector

$$x = A^{-1}b,$$

provides a solution of the system (1.3).

The most important properties of the ordinary inverse are summarized in the following proposition

Proposition 1.2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a given nonsingular matrix, then it holds:*
(a) $(A^{-1})^{-1} = A$;
(b) $(A^*)^{-1} = \left(A^{-1}\right)^*$;
(c) $(AB)^{-1} = B^{-1}A^{-1}$;
(d) *A vector x is an eigenvector of A corresponding to the eigenvalue $\lambda \neq 0$ if and only if x is an eigenvector of A^{-1} corresponding to the eigenvalue λ^{-1}.*
(e) *A vector x is a λ-vector of A of grade p if and only if x is a λ^{-1}-vector of A^{-1} of grade p.*

According to adoptive notation, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\text{rank}(A)$ and A^* denote the null space, the range (column space), the rank and the conjugate transpose, respectively, of $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices. Further, $\mathbb{C}_r^{m\times n} = \left\{X \mid X \in \mathbb{C}^{m\times n}, \text{rank}(X) = r\right\}$. For two complementary subspaces N and M of $\mathbb{C}^{m\times 1}$, $P_{N,M}$ represents a projector onto N along M. The orthogonal projector onto N is denoted by P_N.

The idea for introducing the definition of generalized inverses of matrices arisen from the necessity of finding a solution of a given system of linear equations (SoLE). This problem appears in many scientific and practical disciplines, such as: statistics, operational research, physics, economy, electrical engineering, and many others. Generalized inverses provide a simple way for obtaining a solution of the so called ill-conditioned linear problems. Explicitly, generalized inverses of matrices has appeared bit later in 1920 in the paper [92] of the scientist Moore. However, his work was not continued in the next 30 years, first of all because of the way the work was presented and the ambiguous notation. The research on this topic was initiated by the scientist Bjerhammar in 1951. The real evolution in the development of this area has started with the paper [121] published by Penrose in 1955. The general reciprocal, originated by Bjerhammar and rediscovered by Penrose in [122, 121] is known as the Moore-Penrose inverse and reached enormous popularity.

For arbitrary $A \in \mathbb{C}^{m\times n}$, there exists the Moore-Penrose (or shortly MP) inverse of A (denoted by $A^\dagger$), that is, the unique matrix $X \in \mathbb{C}^{n\times m}$ which satisfies the *Penrose equations* [121]

$$\begin{array}{cc} (1) \quad AXA = A, & (2) \quad XAX = X, \\ (3) \quad (AX)^* = AX, & (4) \quad (XA)^* = XA. \end{array} \tag{1.4}$$

The symbol $A\{\rho\}$ is stated for the set of all matrices that satisfy equations involved in $\rho \subseteq \{1,2,3,4\}$. A ρ-inverse of A, marked with $A^{(\rho)}$, is any matrix from $A\{\rho\}$. Notice that $A\{1,2,3,4\} = \{A^\dagger\}$. If A is a square regular matrix, then its inverse matrix A^{-1} trivially satisfies the system (1.9), i.e., $A^\dagger = A^{-1}$.

The significance of the Moore-Penrose inverse is confirmed by many theoretical studies and applied to research areas such as singular matrix problems, ill-posed problems, optimization problems, statistics, robotics, digital image restoration [23, 24, 48], physics [3], data encryption [50], in finding the the Moore-Penrose solution of the portfolio optimization problem [72], in finding exact Moore-Penrose inverse solutions to fuzzy linear systems [91], in sampling theory related with the problem of signal reconstruction [1], or in kinematic synthesis of the constant transmission ratio spatial linkage [120].

Penrose and later many researches have used this generalized inverse for problems such as solving systems of linear and matrix equations as well as in finding a new type of spectral decomposition. Our important interest in this paper arises from the Penrose's paper [122], published in 1956 and known as *best approximation solutions of linear matrix equations*, which means that $\|Ax - b\| \geq \|AA^\dagger b - b\|$ and $\|A^\dagger b\| < \|x\|$ for all $x \neq A^\dagger b$ and $Ax = b$, for any $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. This very well known result exploits the generalized inverse of a matrix to find the best approximate solution x to the matrix equation $Ax = b$, where A is rectangular and non-square or square and singular.

An *inner inverse* (or $\{1\}$-inverse) of A is a matrix $X \in \mathbb{C}^{n\times m}$ such that $AXA = A$ holds. A particular inner inverse is denoted by $A^{(1)}$. The set of all inner inverses of A will be denoted by $A\{1\}$. An *outer inverse* (or $\{2\}$-inverse) of A is a matrix $X \in \mathbb{C}^{n\times m}$ which satisfies the equation $XAX = X$. A particular outer inverse is denoted by $A^{(2)}$.

The outer inverses of A with determined null space and range attracted attention of many authors because of their uniqueness. Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. The outer inverse of A with the range T and the null-space S (denoted by $A^{(2)}_{T,S}$) is a matrix $X \in \mathbb{C}^{n\times m}$ such that

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S.$$

Recall that A has the outer inverse X satisfying $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$, and in this case $X := A^{(2)}_{T,S}$ is unique. The notation $A \in \mathbb{C}^{m\times n}_{T,S}$ will indicate that $A \in \mathbb{C}^{m\times n}$ and $A^{(2)}_{T,S}$ exists. Notice that

$$A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}.$$

Further abstraction leads to $A\{\mathfrak{I}\}_{T,*} = \{X \in A\{\mathfrak{I}\}| \ \mathcal{R}(X) = T\}$ (resp. $A\{\mathfrak{I}\}_{*,S} = \{X \in A\{\mathfrak{I}\}| \ \mathcal{N}(X) = S\}$) that comprises $\{\mathfrak{I}\}$-inverses of A with known only range T (resp. only kernel S). Finally, $A\{\mathfrak{I}\}_{T,S}$ denotes the set of $\{\mathfrak{I}\}$-inverses of A possessing range T and kernel S.

The outer inverses have many applications in statistics [45, 126], in the iterative themes for tackling nonlinear equations [8], in stable approximations of ill-posed problems and in linear and nonlinear issues implicating rank-deficient generalized inverses [118]. Many interesting results were considered outer inverses [9, 20, 140, 172, 176, 175].

Consider that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$. An outer inverse of A with prescribed range $\mathcal{R}(B)$ (denoted by $A^{(2)}_{\mathcal{R}(B),*}$) is a solution to the following constrained equation:

$$XAX = X, \qquad \mathcal{R}(X) = \mathcal{R}(B). \tag{1.5}$$

The class of outer inverses with the predefined range $\mathcal{R}(B)$ is denoted by $A\{2\}_{\mathcal{R}(B),*}$. Further, an outer inverse of A with prescribed kernel $\mathcal{N}(C)$ (denoted by $A^{(2)}_{*,\mathcal{N}(C)}$) is a solution to the following constrained equation:

$$XAX = X, \qquad \mathcal{N}(X) = \mathcal{N}(C). \tag{1.6}$$

The symbol $A\{2\}_{*,\mathcal{N}(C)}$ will be stand for the class of outer inverses with the predefined kernel $\mathcal{N}(C)$. Last, an outer inverse of A with prescribed range $\mathcal{R}(B)$ and kernel $\mathcal{N}(C)$ (denoted by $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$) is the unique solution of the following constrained equation:

$$XAX = X, \qquad \mathcal{R}(X) = \mathcal{R}(B), \qquad \mathcal{N}(X) = \mathcal{N}(C). \tag{1.7}$$

The key characterizations, representations and computational procedures for outer inverses with prescribed range and/or kernel were discovered in [14, 161, 172, 179, 182] and other research articles cited in these references. More details can be found in the monographs [8, 156, 166]. Full rank representations of outer inverses are given in [129, 130]. Characterizations, representations and computational procedures based on appropriate matrix equations and ranks of involved matrices are proposed in [132, 133, 134]. Iterative computational algorithms were developed in [26, 28, 54, 83, 142].

For $A \in \mathbb{C}^{n\times n}$, there exists the *Drazin inverse* of A (denoted by A^{D}), i.e., the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

where $k = \mathrm{ind}(A)$ is the index of A, i.e., the smallest nonnegative integer k for which the equality $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$ holds. In the case $\mathrm{ind}(A) = 1$ the Drazin inverse A^{D} becomes the *group inverse* of A (denoted by $A^{\#}$). It is known that

$$A^{\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} \qquad \text{and} \qquad A^{\#} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

1.3 Properties of main generalized inverses

The Kronecker product of two matrices is very important in matrix theory.

Let $A=[a_{ij}]_{i=\overline{1,m},j=\overline{1,n}}\in\mathbb{C}^{m\times n}$ be given matrix. By $a=\mathrm{vec}(A)\in\mathbb{C}^{mn}$ is denoted the vector obtained by arranging the elements of A by rows.

Definition 1.3.1. *The Kronecker product $A \otimes B$ of two matrices $A = [a_{ij}] \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ is the $mp \times nq$ matrix expressible in partitioned form as*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

The properties of the Kronecker product are summarized in the following proposition.

Proposition 1.3.1. *Let A, B, E, F be matrices of appropriate dimensions. Then the following hold:*
(a) $(A \otimes B)(E \otimes F) = AE \otimes BF$,
(b) *For any $q \in \mathbb{N}$ it holds $(A \otimes I)^q = A^q \otimes I$.*
(c) *If $\mathrm{ind}(A) = k$, then $\mathrm{ind}(A \otimes I) = k$.*
(d) *If A is a square nonsingular matrix, then the matrix $A \otimes I$ is nonsingular and $(A \otimes I)^{-1} = A^{-1} \otimes I$.*

An important application of the Kronecker product is rewriting a matrix equation

$$AXB = D, \tag{1.8}$$

as a vector equation of the form

$$(A \otimes B^T)\text{vec}(X) = \text{vec}(D).$$

For simplicity, further in text, we denote $A_B = A \otimes B$.

1.3.1 Matrix equations and $\{i, j, \dots, k\}$-inverses

Many problems that usually arise in practice reduce to a problem of the type (1.3), where the matrix A is singular, and moreover, in many cases it is not even a square matrix. The notion of generalized inverse of a matrix is defined to overcome the previous problem.

It is well known the fact that the system (1.3), always has at least one solution if and only if $b \in \mathcal{R}(A)$. This fact means that $b \in \mathcal{R}(A)$ if and only if there exists a matrix X such that $x = Xb$ is a solution of the system. In order to describe such a matrix, there were established the four so called Penrose equations [122]:

$$\begin{array}{ll} (1) & AXA = A \ \ \text{(general condition)} \\ (2) & XAX = X \ \ \text{(reflexive condition)} \\ (3) & (AX)^* = AX \ \ \text{(normalized condition)} \\ (4) & (XA)^* = XA \ \ \text{(reversed normalized condition).} \end{array} \tag{1.9}$$

Proposition 1.3.2. (Penrose 1955) [122] *For any matrix $A \in \mathbb{R}^{m\times n}$, the system of matrix equations $(1), (2), (3), (4)$ in (1.9) has the unique solution $X \in \mathbb{R}^{n\times m}$. This solution is known as the Moore-Penrose (M-P) inverse (pseudoinverse) of A and denoted by $A^\dagger$.*

If A is a square regular matrix, then its inverse matrix A^{-1} trivially satisfies the system (1.9). This means that the Moore-Penrose inverse of a nonsingular matrix is the same as its ordinary inverse, i.e., $A^\dagger = A^{-1}$. Equations (1.9) are called *Penrose equations* and they are used for deriving various classes of generalized inverses. The generalized inverses which satisfy some of above mentioned equations are useful. For a subset $\mathcal{S}$ of the set $\{1, 2, 3, 4\}$, the set of all matrices obeying the equations among $(1), \dots, (4)$ from (1.9) which are defined by $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. Any matrix from $A\{\mathcal{S}\}$ is called $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$.

In this way we come to the notion of $\{i, j, \dots, k\}$-inverses, where $i, j, k \in \mathcal{S}$. For example, for a given matrix $A \in \mathbb{C}^{m\times n}$, if there exists a matrix such that it satisfies only the first Penrose equation, then this matrix is called an $\{1\}$-inverse of the matrix A and it is denoted by $A^{(1)}$. Similarly, if the generalized inverse satisfies the first and the third Penrose equations, it is an $\{1, 3\}$-inverse of A, denoted by $A^{(1,3)}$ while the corresponding set is denoted by $A\{1, 3\}$.

For a given subspaces T and S from $\mathbb{C}^n$ by $P_{T,S}$ we denote a projector from $\mathbb{C}^n$ on T along S. If $S = T^\perp$, i.e., if S is orthogonal complement of T, then P_T is orthogonal projector from $\mathbb{C}^n$ on T. The matrix which corresponds to a linear map which is a projector, is idempotent matrix. The matrix which corresponds to a linear map which is an orthogonal projector is a Hermitian idempotent matrix. In the sequel, we restate the main properties of $\{i, j, \dots, k\}$-inverses, without proofs. For more details, see also [8, 156].

Properties and representations of $\{1\}$-inverses

Lemma 1.3.1. *Let $A \in \mathbb{C}_r^{m\times n}$, and let $E \in \mathbb{C}_m^{m\times m}$ and $P \in \mathbb{C}_n^{n\times n}$ be matrices satisfying*

$$EAP = \begin{bmatrix} I_r & K \\ O & O \end{bmatrix}.$$

Then the $n \times m$ matrix

$$X = P \begin{bmatrix} I_r & K \\ O & L \end{bmatrix} E \tag{1.10}$$

is an $\{1\}$-inverse (or inner inverse) of A, for any $L \in \mathbb{C}_m^{(n-r)\times(m-r)}$.

Lemma 1.3.2. *For a given matrix $A \in \mathbb{C}_r^{m\times n}$ the following statement are valid:*

(a) $(\lambda A)^{(1)} = \lambda^\dagger A^{(1)}$, *where $\lambda \in \mathbb{C}$ and* $\lambda^\dagger = \left\{ \begin{array}{ll} \frac{1}{\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{array} \right.$;

(b) $AA^{(1)}$ *is a projection from $\mathbb{C}^m$ on $\mathcal{R}(A)$, i.e., $AA^{(1)} = P_{\mathcal{R}(A),S}$ where $S \in \mathbb{C}^m$ is such that $\mathcal{R}(A) + S = \mathbb{C}^m$;*

(c) $I - A^{(1)}A$ *is a projection from* $\mathbb{C}^n$ *on* $\mathcal{N}(A)$*, i.e.,* $I - A^{(1)}A = P_{\mathcal{N}(A),T}$ *where* $T \in \mathbb{C}^n$ *is such that* $T + \mathcal{N}(A) = \mathbb{C}^n$*;*

(d) $\operatorname{rank}(A^{(1)}) \geq \operatorname{rank}(A)$*;*

(e) $A^{(1)}A = I_n$ *if and only if* $r = n$*;*

(f) $AA^{(1)} = I_m$ *if and only if* $r = m$*;*

(g) *If* $X \in A\{1\}$*, then* $X \in A\{1,2\}$ *if and only if* $\operatorname{rank}(A) = \operatorname{rank}(X)$*;*

(h) $(A^*A)^{(1)}A^* \in A\{1,2,3\}$*;*

(i) $A^*(AA^*)^{(1)} \in A\{1,2,4\}$*;*

(j) $A^{(1,4)}AA^{(1,3)} = A^\dagger$.

The next results establish the extremely important relationship between $\{i, j, \ldots, k\}$-inverses and the solutions of a linear matrix equation [8, 156].

Lemma 1.3.3. *Let* $A \in \mathbb{C}^{m\times n}$*,* $B \in \mathbb{C}^{p\times q}$*,* $D \in \mathbb{C}^{m\times q}$*. Then the matrix equation*

$$AXB = D$$

is consistent if and only if it holds

$$AA^{(1)}DB^{(1)}B = D,$$

for some $A^{(1)}, B^{(1)}$*. In this case, the general solution is*

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)}$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$*.*

Corollary 1.3.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $A^{(1)} \in A\{1\}$*. Then*

$$A\{1\} = \{A^{(1)} + Z - A^{(1)}AZAA^{(1)} \mid Z \in \mathbb{C}^{n\times m}\}.$$

Corollary 1.3.2. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $b \in \mathbb{C}^m$*. Then the system* (1.3) *is consistent if and only if for some* $A^{(1)}$ *it holds*

$$AA^{(1)}b = b,$$

in which case the general solution of the system (1.3) *is*

$$x = A^{(1)}b + (I - A^{(1)}A)y,$$

for arbitrary $y \in \mathbb{C}^n$*.*

Lemma 1.3.4. *The matrix equations*

$$AX = B, \quad XD = E$$

have a common solution if and only if each equation separately has a solution, i.e.,

$$AA^{(1)}B = B, \qquad ED^{(1)}D = E, \tag{1.11}$$

and

$$AE = BD.$$

In this case,

$$X = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)}$$

is a common solution of both equations, for arbitrary $A^{(1)}$ *and* $D^{(1)}$*.*

Lemma 1.3.5. *Let the equations given in* (1.11) *have a common solution* $X_0 \in \mathbb{C}^{m\times n}$*. Then the general solution of these equations is given by*

$$X = X_0 + (I - A^{(1)}A)Y(I - DD^{(1)}),$$

for arbitrary $A^{(1)} \in A\{1\}$*,* $D^{(1)} \in D\{1\}$*,* $Y \in \mathbb{C}^{m\times n}$*.*

Proposition 1.3.3. *Let* $A \in \mathbb{C}^{m\times n}$*,* $X \in \mathbb{C}^{n\times m}$*. Then* $X \in A\{1\}$ *if and only if, for all* $b \in \mathcal{R}(A)$*,* $x = Xb$ *is a solution of the system* (1.3)*.*

Proposition 1.3.4. *The identity* $AB(AB)^{(1)}A = A$ *holds if and only if* $\operatorname{rank}(AB) = \operatorname{rank}(A)$*. Similarly,* $B(AB)^{(1)}AB = B$ *is valid if and only if* $\operatorname{rank}(AB) = \operatorname{rank}(B)$*.*

Proposition 1.3.5. *Let* $A \in \mathbb{C}^{m\times n}$ *and let* $A^{(1)}$ *be an arbitrary element of* $A\{1\}$*. Further, denote by* $\mathcal{R}(A) = L$ *and* $\mathcal{N}(A) = M$*. Then* $AA^{(1)}$ *and* $A^{(1)}A$ *are idempotent and*

$$AA^{(1)} = P_{L,S}, \quad A^{(1)}A = P_{T,M},$$

where S *is a subspace of* $\mathbb{C}^m$ *complementary to* L*, and* T *is a subspace of* $\mathbb{C}^n$ *complementary to* M*.*

Properties and representations of $\{1,2\}$-inverses

It is known that the existence of a $\{1\}$-inverse of a matrix A implies the existence of its $\{1,2\}$-inverse. This fact is verified in Lemma 1.3.6.

Lemma 1.3.6. *Let $Y, Z \in A\{1\}$. Then $X = YAZ \in A\{1,2\}$.*

According to Lemma 1.3.6, for any $L \in \mathbb{C}^{(n-r)\times(m-r)}$, the $n \times m$ matrix X defined in (1.10) belongs to $A\{1,2\}$ if and only if X is given in the form (1.10).

Lemma 1.3.7. (Bjerhammar 1958) [**?**] *For a given A and $X \in A\{1\}$, it follows that $X \in A\{1,2\}$ if and only if* $\operatorname{rank}(X) = \operatorname{rank}(A)$.

Lemma 1.3.8. *Any two of the following three statements imply the third:*

$$\begin{gathered} X \in A\{1\}, \\ X \in A\{2\}, \\ \operatorname{rank}(X) = \operatorname{rank}(A). \end{gathered}$$

Proposition 1.3.6. *If A and X are $\{1,2\}$-inverses of each other, then*

$$AX = P_{\mathcal{R}(A),\mathcal{N}(X)}, \quad XA = P_{\mathcal{R}(X),\mathcal{N}(A)}.$$

Properties and representations of $\{1,3\}$, $\{1,4\}$, $\{1,2,3\}$ and $\{1,2,4\}$-inverses

Urquhart in [150] has shown that the existence of a $\{1\}$-inverse of every finite complex matrix A implies the existence of an $\{1,2,3\}$-inverse and an $\{1,2,4\}$-inverse of A. This result is restated in Lemma 1.3.9.

Lemma 1.3.9. (Urquhart 1968) [150]. *For every finite complex matrix A,*

$$Y = (A^*A)^{(1)}A^* \in A\{1,2,3\}$$

and

$$Z = A^*(AA^*)^{(1)}A \in A\{1,2,4\}.$$

Proposition 1.3.7. *The set $A\{1,3\}$ consists of all solutions X of the system*

$$AX = AA^{(1,3)},$$

where $A^{(1,3)}$ is an arbitrary element of $A\{1,3\}$.

Proposition 1.3.8. *Let $A \in \mathbb{C}^{m\times n}$ and $A^{(1,3)} \in A\{1,3\}$. Then*

$$A\{1,3\} = \{A^{(1,3)} + (I - A^{(1,3)}A)Z \mid Z \in \mathbb{C}^{n\times m}\}.$$

Proposition 1.3.9. *The set $A\{1,4\}$ consists of all solutions X of the matrix equation*

$$XA = A^{(1,4)}A,$$

where $A^{(1,4)}$ is an arbitrary element of $A\{1,4\}$.

Corollary 1.3.3. *Let $A \in \mathbb{C}^{m\times n}$ and $A^{(1,4)} \in A\{1,4\}$. Then*

$$A\{1,4\} = \{A^{(1,4)} + Y(I - AA^{(1,4)}) \mid Y \in \mathbb{C}^{n\times m}\}.$$

Generalized inverses give us a powerful tool for finding a solution of a consistent system of linear equation. Moreover, the generalized inverses are useful for finding an approximate solution of an inconsistent system of linear equations.

1.3.2 Basic properties of the Moore-Penrose inverse

The most important result related to the Penrose equations is the statement that there always exists a unique matrix which satisfies the four Penrose equations. This result was shown by Penrose [122] in 1955. This matrix is called the Moore-Penrose inverse and denoted by $A^\dagger$.

The concept of a generalized inverses of an arbitrary matrix $A \in \mathbb{C}^{m\times n}$ is originally due to Moore, in 1920, (called by him the "general reciprocal"). His definition was essentially as follows.

Definition 1.3.2. *If $A \in \mathbb{C}^{m\times n}$, then the generalized inverse of A is the matrix $X \in \mathbb{C}^{n\times m}$ such that*

$$1.\quad AX = P_{\mathcal{R}(A)}; \qquad 2.\quad XA = P_{\mathcal{R}(X)}. \tag{1.12}$$

Moore in [92] proved the existence and the uniqueness of the solution of such defined generalized inverse by proving the following result.

Proposition 1.3.10. *For every $A \in \mathbb{C}^{m\times n}$ there exists a unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying* (1.12).

Rado proved the equivalence of Moore's and Penrose's definitions of the generalized inverse, and today this inverse is known as Moore-Penrose pseudoinverse (shortly M-P inverse or pseudoinverse).

Although $\{1\}$-inverses and $\{1,3\}$-inverses provide a solution of a given matrix equation, the Moore-Penrose inverse most resemble to the ordinary inverse. This statement is justified by its uniqueness and the properties listed in the following two lemmas. Also, since the Moore-Penrose inverse is $\{1\}$-inverse, we should take into account that the properties from Lemma 1.3.2 are also valid for the Moore-Penrose inverse.

Proposition 1.3.11. (Penrose 1955) [122] *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^{m\times 1}$. The minimal-norm least-squares solution of the system $Ax = b$ is given by $x^* = A^\dagger b$. All other least-squares solutions are given by*

$$x = A^\dagger b + (I_n - A^\dagger A)z, \quad z \in \mathbb{C}^n.$$

Lemma 1.3.10. *Let $A \in \mathbb{C}^{m\times n}$ be an arbitrary matrix. The Moore-Penrose inverse $A^\dagger$ possesses the following properties:*

(a) $(A^\dagger)^\dagger = A$, $(A^\dagger)^* = (A^*)^\dagger$;

(b) $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$, $(A^*A)^\dagger = A^\dagger (A^*)^\dagger$;

(c) $A^\dagger AA^* = A^* = A^*AA^\dagger$;

(d) $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$;

(e) $\mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)$

(f) $\mathcal{R}(AA^*) = \mathcal{R}(AA^{(1)}) = \mathcal{R}(A)$, $\mathrm{rank}(AA^{(1)}) = \mathrm{rank}(A^{(1)}A) = \mathrm{rank}(A)$;

(g) $AA^\dagger = P_{\mathcal{R}(A^*),\mathcal{N}(A)}$ *and* $A^\dagger A = P_{\mathcal{R}(A),\mathcal{N}(A^*)}$.

Lemma 1.3.11. *Let $A \in \mathbb{C}^{m\times n}$ be an arbitrary matrix. Then the matrix A can be written in the form*

$$A \sim \begin{bmatrix} A_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{1.13}$$

where A_1 is invertible. Hence,

$$A^\dagger \sim \begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The representation (1.13) of can be easily obtained from the Singular value decomposition (SVD) of A. More precisely, the SVD decomposition of A assumes that the matrix A_1 is a diagonal matrix whose entries are the singular values of A.

If the vector b in the system (1.3) satisfies $b \notin \mathcal{R}(A)$, then it is necessary to search for an approximate solution by trying to find a vector x which minimizes the norm of the vector $Ax - b$.

Definition 1.3.3. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. A vector $\hat{x} \in \mathbb{C}^n$ which satisfies the minimization problem*

$$\|A\hat{x} - b\|^2 = \min_{x\in\mathbb{C}^n} \|Ax - b\|^2. \tag{1.14}$$

is called a least-squares solution *of the system* (1.3).

The next lemma gives a characterization of all least-squares solutions of the system (1.3).

Lemma 1.3.12. *The vector x is a least-squares solution of the system* (1.3) *if and only if x is a solution of the normal equation, defined by*

$$A^*Ax = A^*b. \tag{1.15}$$

The following proposition, restated from [8], shows that $\|Ax-b\|$ is minimized by the vector $x = A^{(1,3)}b$. This statement establishes very important relation between the set of $\{1,3\}$-inverses and the least-squares solutions of the system (1.3).

Proposition 1.3.12. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then $\|Ax - b\|$ is smallest when $x = A^{(1,3)}b$, where $A^{(1,3)} \in A\{1,3\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, $\|Ax - b\|$ is smallest when $x = Xb$, then $X \in A\{1,3\}$.*

Since $A^{(1,3)}$ inverse of a matrix is not unique, as a consequence, a system of linear equations has many least-squares solutions in general. However, among all least-squares solutions of a given system of linear equations, there exists only one such solution of minimum norm.

Definition 1.3.4. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. A vector $\hat{x}$, which satisfies the minimization problem*

$$\|\hat{x}\|^2 = \min_{x\in\mathbb{C}^n} \|x\|^2 \tag{1.16}$$

is called a minimal-norm solution *of the system* (1.3)*.*

The next proposition, restated from [8], establishes a relation between $\{1,4\}$-inverses and the minimum-norm solutions of the system (1.3).

Proposition 1.3.13. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. If $Ax = b$ has a solution for x, the unique solution x for which $\|x\|$ is smallest is given by $x = A^{(1,4)}b$, where $A^{(1,4)} \in A\{1,4\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ is such that, whenever $Ax = b$ has a solution, $x = Xb$ is the solution of minimal-norm, then $X \in A\{1,4\}$.*

Joining the results from Proposition 3.1.1 and Proposition 3.1.2 we are coming to the most important property of the Moore-Penrose inverse.

Corollary 1.3.4. (Penrose 1955) [122] *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then, among the least-squares solutions of $Ax = b$, $A^\dagger b$ is the one of minimum-norm. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, Xb is the* minimal-norm least-squares solution *of $Ax = b$, then $X = A^\dagger$.*

In the essence, Lemma 3.1.1 shows that $A^\dagger b$ is the minimal-norm least-squares solution of the linear system $Ax = b$. This fact caused a dramatic increase of the interest in the generalized inverses theory.
Further, the next proposition characterizes the set of all least-squares solutions of a given system of linear equations.

Proposition 1.3.14. (Nashed 1970, 1976) [119, 118] If $A \in \mathbb{C}^{m\times n}$ has a closed range $\mathcal{R}(A)$, then the set S of all least-squares solutions of the system $Ax = b$ is given by

$$S = A^\dagger b \oplus \mathcal{N}(A) = \{A^\dagger b + (I - A^\dagger A)y \mid y \in \mathbb{C}^n\},$$

where $\mathcal{N}(A)$ denotes the null space of A.

Some additional properties of $A^\dagger$ and $A^{(1)}$ can be found for example in [8, 156].
The Moore-Penrose inverse can be computed using arbitrary $\{1\}$-inverse.

Proposition 1.3.15. (Yanai, Takeuchi, Takane 2011) [178] *The Moore-Penrose inverse $A^\dagger$ can be expressed by an arbitrary inner inverse, as*

$$\begin{aligned} A^\dagger &= A^{\mathrm{T}}A\left(A^{\mathrm{T}}AA^{\mathrm{T}}A\right)^{(1)}A^{\mathrm{T}} \\ &= A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{(1)}A\left(A^{\mathrm{T}}A\right)^{(1)}A^{\mathrm{T}}. \end{aligned}$$

In the following three statements we will give the representation of $A^\dagger$ using the SVD decomposition and the full rank factorization.

Lemma 1.3.13. *Let $A \in \mathbb{C}^{m\times n}$ be arbitrary matrix. Consider the decomposition $A = Q^*RP$. Then the Moore-Penrose inverses of matrices R and A, $R^\dagger$ and $A^\dagger$ can be represented by*

$$R^\dagger = \begin{bmatrix} R_{11}^{-1} & O \\ O & O \end{bmatrix}, \quad A^\dagger = P^*R^\dagger Q. \tag{1.17}$$

The special case of Lemma 1.3.13 is representation of $A^\dagger$ by the singular value decomposition of the matrix A.

Lemma 1.3.14. *Let $A \in \mathbb{C}_r^{m\times n}$ and let $A = U\Sigma V^*$ be the singular value decomposition of A where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices and $A = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\mathrm{r}})$, $\sigma_{\mathrm{i}} = \sqrt{\lambda_{\mathrm{i}}}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ are the nonzero eigenvalues of A^*A. If*

$$A = U \begin{bmatrix} \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\mathrm{r}}) & O \\ O & O \end{bmatrix} V^* = U \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} V^* \in \mathbb{C}^{m\times n},$$

then

$$A^\dagger = V \begin{bmatrix} \mathrm{diag}(1/\sigma_1, 1/\sigma_2, \dots, 1/\sigma_{\mathrm{r}}) & O \\ O & O \end{bmatrix} U^* = V \begin{bmatrix} \Sigma^{-1} & O \\ O & O \end{bmatrix} U^* \in \mathbb{C}^{n\times m}.$$

Moreover, let S_1, S_2, S_3 be arbitrary $r \times m - r$, $n - r \times r$, and $m - r \times n - r$ matrices, respectively. Then an inner inverse of A is given by

$$A^{(1)} = V \begin{bmatrix} \mathrm{diag}(1/\sigma_1, 1/\sigma_2, \dots, 1/\sigma_{\mathrm{r}}) & S_1 \\ S_2 & S_3 \end{bmatrix} U^* \in \mathbb{C}^{n\times m}.$$

Theorem 1.3.1. *Let $A \in \mathbb{C}_r^{m\times n}$, let*

$$A = U \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} V^* \tag{1.18}$$

be the singular value decomposition (SVD decomposition) of A, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$, $\sigma_i = \sqrt{\lambda_i}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ are the nonzero eigenvalues of A^*A. Then $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the nonzero singular value of A and

$$\|A\|_2 = \sigma_1, \; \|A^\dagger\|_2 = \frac{1}{\sigma_r}. \tag{1.19}$$

Proof. From (1.18), we have

$$A^*A = V \begin{bmatrix} \Sigma^2 & O \\ O & O \end{bmatrix} V^*.$$

Thus the eigenvalues of A^*A are $\sigma_i^2 = \lambda_i(A^*A),\ i = 1, 2, \dots, n$ and

$$\|A\|_2^2 = \|A^*A\|_2 = |\lambda_1(A^*A)| = \sigma_1^2.$$

So $\|A\|_2 = \sigma_1$ holds. It is easy to verify that

$$A^\dagger = V \begin{bmatrix} \Sigma^{-1} & O \\ O & O \end{bmatrix} U^*. \tag{1.20}$$

Hence the non-zero singular values of $A^\dagger$ are

$$\frac{1}{\sigma_r} \geq \frac{1}{\sigma_{r-1}} \geq \cdots \geq \frac{1}{\sigma_1} > 0.$$

Thus $\|A^\dagger\|_2 = \dfrac{1}{\sigma_r}$ holds. □

Next lemma shows that the full rank factorization of a matrix A leads to an explicit formula for its Moore-Penrose inverse $A^\dagger$. This formula is known as the full rank representation of the Moore-Penrose inverse.

As usual, by A_R^{-1} and A_L^{-1} we denote a right and a left inverse of A, respectively. If a matrix A is of dimensions $m \times n$ and of rank $raA = m$, then there exists an $n \times m$ matrix A_R^{-1} called the right inverse of A satisfying $AA_R^{-1} = I_m$. If a matrix A is of dimensions $m \times n$ and of rank $raA = n$, then there exists an $n \times m$ matrix A_R^{-1} called the left inverse of A satisfying $A_L^{-1}A = I_n$.

Lemma 1.3.15. (MacDuffe, 1956) [87] *Let $A \in \mathbb{C}_r^{m\times n}$ and $A = PQ$, $P \in \mathbb{C}_r^{m\times r}$, $Q \in \mathbb{C}_r^{r\times n}$ be its full rank factorization. Then it holds*

$$A^\dagger = Q^*(P^*AQ^*)^{-1}P^* = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*.$$

In addition, Q is right invertible and P is left invertible:

$$Q_R^{-1} = Q^*(QQ^*)^{-1}, \quad P_L^{-1} = (P^*P)^{-1}P^*.$$

1.3.3 Basic properties of the weighted Moore-Penrose inverse

The relationships between the generalized inverses $A^{(1,4)}$, $A^{(1,3)}$, $A^{\dagger}$ and the minimal-norm solution, least-squares solution and minimum-norm least-squares solution are discussed in the previous subsection. In these cases, minimization was considered with respect to the usual inner product $\langle x, y\rangle = y^*x$ and the vector norm $\|x\| = \langle x, x\rangle^{1/2}$. In general, it is possible to study different weighted norms for the solution x and for the residual $Ax - b$ of the linear system $Ax = b$.

Let $A \in \mathbb{C}^{m\times n}$ and let $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be Hermitian positive definite matrices. Weighted inner products in spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ can be defined by

$$\langle x, y\rangle_M = y^*Mx, \quad \langle x, y\rangle_N = y^*Nx.$$

According to these scalar products, the weighted norm $\|x\|_M^2 = \langle x, x\rangle_M = x^*Mx$ can be defined as usual. Let us recall that the conjugate transpose matrix A^* satisfies $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for every $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. The *weighted conjugate transpose matrix* $A^\sharp$ can be introduced in the same manner.

Lemma 1.3.16. *For every Hermitian positive definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ and an arbitrary matrix $A \in \mathbb{C}^{m\times n}$, there exists a unique* weighted conjugate transpose matrix $A^\sharp = N^{-1}A^*M$ *which satisfies $\langle Ax, y\rangle_M = \langle x, A^\sharp\rangle_N$.*

The relations between the weighted generalized inverses and the solutions of linear equations are given as follows.

Theorem 1.3.2. *Let $A \in \mathbb{C}^{m\times n}$ and N be a Hermitian positive definite matrix of the order n. Then $x = Xb$ is the minimal-norm solution (with respect to the norm $\| \; \|_N$) of the consistent system of linear equations $Ax = b$ for any $b \in R(A)$ if and only if X satisfies*

$$(1) \quad AXA = A \qquad (4N) \quad (NXA)^* = NXA. \tag{1.21}$$

Every matrix X satisfying (1.21) is called $\{1, 4N\}$ inverse and denoted by $A^{(1,4N)}$. The set of all $A^{(1,4N)}$ inverses is denoted by $A\{1, 4N\}$.

Theorem 1.3.3. *Let $A \in \mathbb{C}^{m\times n}$ and M be a Hermitian positive defnite matrix of order m. Then $x = Xb$ is the least-squares (according to the norm $\| \; \|_M$) solution of the inconsistent system of linear equations $Ax = b$ for every $b \notin R(A)$ if and only if X satisfies*

$$(1) \quad AXA = A \quad (3M) \quad (MAX)^* = MAX. \tag{1.22}$$

Every matrix X satisfying (1.22) is called $\{1, 3M\}$ inverse and denoted by $A^{(1,3M)}$. As usual, the set of all $A^{(1,3M)}$ inverses is denoted by $A\{1, 3M\}$.

Theorem 1.3.4. *Let $A \in \mathbb{C}^{m\times n}$ and M, N be Hermitian positive defnite matrices of orders m and n respectively. Then $x = Xb$ is the weighted minimal-norm (with respect to the norm $\| \; \|_N$) least-squares (according to the norm $\| \; \|_M$) solution of the inconsistent system of linear equations $Ax=b$ for every $b \notin R(A)$ if and only if X satisfies*

$$\begin{array}{ll} (1)\ AXA = A & (2)\ XAX = X \\ (3M)\ (MAX)^* = MAX & (4N)\ (NXA)^* = NXA. \end{array} \tag{1.23}$$

Moreover, system of matrix equations (1.23) *has a unique solution.*

A matrix X satisfying (1.23) is called *the weighted Moore-Penrose inverse*, and it is denoted by $X = A^{\dagger}_{MN}$. The weighted Moore-Penrose inverse $A^{\dagger}_{MN}$ is the generalization of Moore-Penrose inverse $A^{\dagger}$. If $M = I_m$, $N = I_n$, then $A^{\dagger}_{MN} = A^{\dagger}$. Basic properties of $A^{\dagger}_{MN}$ are given as follows.

Lemma 1.3.17. *Let $A \in \mathbb{C}^{m\times n}$ and $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ be Hermitian positive definite matrices. Then the weighted Moore-Penrose inverse possesses the following properties:*

(a) $(A^{\dagger}_{MN})^{\dagger}_{NM} = A$,

(b) $(A^{\dagger}_{MN})^* = (A^*)^{\dagger}_{N^{-1}M^{-1}}$,

(c) $A^{\dagger}MN = (A^*MA)^{\dagger}_{I_mN}A^*M = N^{-1}A^*(AN^{-1}A^*)^{\dagger}_{MI_n}$,

(d) *If $A = FG$ is a full rank factorization of A, then* $A^{\dagger}_{MN} = N^{-1}G^*(F^*MAN^{-1}G^*)^{-1}F^*M$,

(e) $A^{\dagger}_{MN} = N^{-1/2}(M^{1/2}AN^{-1/2})^{\dagger}M^{1/2}$.

For more details about the weighted Moore-Penrose inverse see [12, p.118, exercise 30], or [149, Sect. 3]. For computational methods on generalized inverses see the monograph [167], and for more on this subject, see [36, 37]. Also, it is also known (see e.g., [8]) that

$$A^{\dagger}_{M,N} = N^{-\frac{1}{2}} \left(M^{\frac{1}{2}} A N^{-\frac{1}{2}} \right)^{\dagger} M^{\frac{1}{2}}.$$

In this case, $A^{\dagger}_{M,N}b$ is the M-least squares solution of $Ax = b$ which has the minimal N-norm.

The notion of the weighted Moore-Penrose inverse can be extended in the case when M and N are positive semidefinite matrices: in this case, the matrix X is such that Xb is a minimal N semi-norm, M-least squares solution of $Ax = b$. When N is positive definite, then there exists a unique solution for X.

Theorem 1.3.5. *Let $A \in \mathbb{C}^{m\times n}_r$, M and N be Hermitian positive definite matrices of orders m and n respectively. Let $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ satisfy*

$$U^* M U = I_m, \qquad V^* N^{-1} V = I_n. \tag{1.24}$$

Assume that A is decomposed in the form

$$A = U \begin{bmatrix} D & O \\ O & O \end{bmatrix} V^*, \tag{1.25}$$

where $D = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_r)$, $\mu_i = \sqrt{\lambda_i}$ and $\lambda_1 \geq \cdots \geq \lambda_r > 0$ are the nonzero eigenvalues of $A^\sharp A = (N^{-1}A^* M)A$. Then $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0$ are the nonzero (M, N) singular values of A, and

$$\|A\|_{MN} = \mu_1, \qquad \|A^{\dagger}_{MN}\|_{NM} = \frac{1}{\mu_r}. \tag{1.26}$$

Proof. By using (1.25), it is easy to verify that

$$A^{\dagger}_{MN} = N^{-1} V \begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix} U^* M. \tag{1.27}$$

Hence the non-zero (M, N) singular values of $A^{\dagger}_{MN}$ are

$$\frac{1}{\mu_r} \geq \frac{1}{\mu_{r-1}} \geq \cdots \geq \frac{1}{\mu_1} > 0.$$

Thus, $\|A^{\dagger}_{MN}\|_{NM} = \dfrac{1}{\mu_r}$. $\square$

1.3.4 Definition and basic properties of the Drazin inverse

It is known that the Moore-Penrose inverse is a very good substitution for the ordinary inverse, when a solution of a given matrix equation is needed. It exists for all matrices, and when the matrix is nonsingular it reduces to the ordinary inverse. But, unfortunately we can not say that it satisfies the properties of the ordinary inverse, characterized with the fourth and fifth item from Lemma 1.2.2. In order to define such inverse, the set of Penrose equations is enlarged with two additional:

(1^p) $\quad A^p X A = A^p,$ $\quad p = \mathrm{ind}(A)$ **(general p condition)**

(5) $\quad AX = XA$ $\quad$ **(commutativity condition)**

Lemma 1.3.18. (Drazin 1958) [34] *Let $A \in \mathbb{C}^{n\times n}$ be arbitrary matrix of index $k = \mathrm{ind}(A)$. Then the following matrix equations*

$$(1^k) \ \ A^k X A = A^k, \qquad (2) \ \ XAX = X, \qquad (5) \ \ AX = XA \tag{1.28}$$

has the unique solution. This solution is called the Drazin inverse of the matrix A and denoted by A^{D}.

The Drazin inverse in the case $p = \mathrm{ind}(A) = 1$ becomes the group inverse, which is denoted by $A^{\#}$.

The main properties of the Drazin inverse are summarized in the next lemma.

Lemma 1.3.19. *Let $A \in \mathbb{C}^{n\times n}$ and $p = \mathrm{ind}(A)$*

(*a*) *$A^l X A = A^l$ for all $l \geq p$.*

(*b*) $\mathcal{R}(A^l) = \mathcal{R}(A^{l+1})$, $\mathcal{N}(A^l) = \mathcal{N}(A^{l+1})$ *and* $\operatorname{rank}(A^l) = \operatorname{rank}(A^{l+1})$, *for all* $l \geq p$. *Moreover, p is the smallest integer for which these equalities hold.*

(*c*) *The matrix A can be written in following way:*

$$A \sim \begin{bmatrix} A_1 & O \\ O & N \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^p) \\ \mathcal{N}(A^p) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^p) \\ \mathcal{N}(A^p) \end{bmatrix}, \tag{1.29}$$

where A_1 is invertible and N is nilpotent matrix. Then

$$A^{\mathrm{D}} \sim \begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^p) \\ \mathcal{N}(A^p) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^p) \\ \mathcal{N}(A^p) \end{bmatrix}. \tag{1.30}$$

(*d*) *For each $\lambda \neq 0$, a vector x is a λ^{-1}-vector of A^{D} of grade s if and only if it is a λ-vector of A of grade s, and x is a 0-vector of A^{D} if and only if it is a 0-vector of A (without regard to its grade).*

Remark 1.3.1. *It is important to mention that the form* (1.29) *of the matrix A, can be obtained from the Jordan decomposition of A. From the computational point of view, that means that the Drazin inverse can be computed using the Jordan decomposition, according to* (1.30).

More general conclusion is that the Singular Value Decomposition is the basis for computing the Moore-Penrose inverse and the Jordan canonical form is appropriate for computing the Drazin inverse.

An exact way for computing the Drazin inverse A^{D} from the Jordan canonical form of the matrix A is described in Theorem 1.32.

Theorem 1.3.6. (Campbell 1979) [12] *Let $A \in \mathbb{C}^{n\times n}$ possesses the Jordan canonical form*

$$A = PJP^{-1} = P \begin{bmatrix} C & O \\ O & N \end{bmatrix} P^{-1}, \tag{1.31}$$

where C is regular and N is nilpotent matrix (there exist an integer k such that $N^k = O$). Then A^{D} possesses the representation

$$A^{\mathrm{D}} = P^{-1} \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P. \tag{1.32}$$

The following lemma restates some basic properties of the Drazin inverse from [34].

Lemma 1.3.20. *Let $A \in \mathbb{C}^{n\times n}$ and let $k = \operatorname{ind}(A)$. Then the following statements hold:*

(a) $(A^*)^{\mathrm{D}} = (A^{\mathrm{D}})^*$,

(b) $(A^n)^{\mathrm{D}} = (A^{\mathrm{D}})^n$ *for any* $n = 1, 2, \ldots$,

(c) $((A^{\mathrm{D}})^{\mathrm{D}})^{\mathrm{D}} = A^{\mathrm{D}}$, $(A^{\mathrm{D}})^{\mathrm{D}} = A$ *if and only if* $k = 1$,

(d) $\mathcal{R}(A^{\mathrm{D}}) = \mathcal{R}(A^l)$ *and* $\mathcal{N}(A^{\mathrm{D}}) = \mathcal{N}(A^l)$ *for every* $l \geq k$,

(e) *If λ is an eigenvalue of A then $\lambda^{\dagger}$ is an eigenvalue of A^{D}.*

Additional properties of the Drazin inverse as well as definitions and properties of other spectral inverses are given, for example, in [8, 156].

Despite the spectral properties, the Drazin inverse in some cases also provides a solution of a given system of linear equations. Namely for $A \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$, as it was shown in [12], $A^{\mathrm{D}}b$ is a solution of the following system

$$Ax = b, \quad \text{where } b \in \mathcal{R}(A^k),\ k = \operatorname{ind}(A). \tag{1.33}$$

The solution $A^{\mathrm{D}}b$ is known as the *Drazin-inverse solution* of the system (3.19). Since the Drazin-inverse provides a solution to the given system in this case, the system (3.19) is termed as *Drazin-consistent system*.

The Drazin inverse has many applications in the theory of finite Markov chains as well as in the study of differential equations and singular linear difference equations [12], cryptography [73] etc.

Establishing a relation between the Drazin inverse and the solutions of a given system of linear equations, naturally imposed the idea of exploring minimal properties of the Drazin inverse. Next we present the results from [171] related to the Drazin-inverse solution.

Theorem 1.3.7. (Wei, Wu 2001) [171] *Let $A \in \mathbb{R}^{n\times n}$ with $p = \mathrm{ind}(A)$. Then $A^{\mathrm{D}}b$ is the unique solution in $\mathcal{R}(A^p)$ of the system*

$$A^{p+1}x = A^p b. \tag{1.34}$$

Theorem 1.3.8. (Wei, Wu 2001) [171] *Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ and $p = \mathrm{ind}(A)$. The set of all solutions of the equation* (3.10) *is given by*

$$x = A^{\mathrm{D}}b + \mathcal{N}(A^p). \tag{1.35}$$

Since the linear system (3.10) is analogous to the normal equation, defined by (1.15), we shall call it the *generalized normal equations* of (3.19).

Let $A = PJP^{-1}$ be the Jordan decomposition of the matrix A. We denote $\|x\|_P = \|P^{-1}x\|$.

Theorem 1.3.9. (Wei, Wu 2001) [171] *Let $A \in \mathbb{R}^{n\times n}$ with $p = \mathrm{ind}(A)$. Then $\hat{x}$ satisfies*

$$\|b - A\hat{x}\|_P = \min_{u\in\mathcal{N}(A)+\mathcal{R}(A^{p-1})} \|b - Ax\|_P$$

if and only if $\hat{x}$ is the solution of the equation

$$A^{p+1}x = A^p b, \quad x \in \mathcal{N}(A) + \mathcal{R}(A^{p-1}).$$

Moreover, the Drazin-inverse solution $x = A^{\mathrm{D}}b$ is the unique minimal P-norm solution of the generalized normal equations (3.10)*.*

Corollary 1.3.5. (Wei, Wu 2001) [171] *Let $A \in \mathbb{C}^{n\times n}$, $p = \mathrm{ind}(A)$ and $b \in \mathcal{R}(A)$. Then the the inequality $\|x\|_P \geq \|A^{\mathrm{D}}b\|_P$ holds for all solutions x of the system* (3.10)*, i.e., $A^{\mathrm{D}}b$ is the unique solution of the equation* (3.10) *of minimum P-norm.*

The notion of the weighted Drazin inverse is an analogy to the weighted Moore-Penrose inverse.

Definition 1.3.5. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then the matrix $X \in \mathbb{C}^{m\times n}$ satisfying*

(a) $(AW)^{p+1}XW = (AW)^p$; *(for some nonnegative integer p)*

(b) $XWAWX = X$

(c) $AWX = XWA$

is called W-weighted Drazin inverse of A, and it is denoted by $X = A_{d,W}$.

1.3.5 Basic properties of outer inverses

Construction of outer inverses of prescribed rank

Recall that, for an arbitrary matrix $A \in \mathbb{C}^{m\times n}$, the set of all outer inverses (or also called $\{2\}$-inverses) is defined by the following set

$$A\{2\} = \{X \in \mathbb{C}^{n\times m} |\ XAX = X\}. \tag{1.36}$$

The set of all outer inverses of rank s is denoted by $A\{2\}_s$, while the notation $A^{(2)}$ stands for an arbitrary outer inverse of A.

Clearly, the rank of an arbitrary outer inverse $A^{(2)}$ satisfies $\mathrm{rank}(A^{(2)}) \leq r = \mathrm{rank}(A)$. We note also that the $n \times m$ null matrix is a $\{2\}$-inverse of rank equal to 0. Also, any element from $A\{1,2\}$ is a $\{2\}$-inverse of A of rank r.

It is possible to generate outer inverses of rank s for an arbitrary integer s between 0 and $r = \mathrm{rank}(A)$. The method is based on full-rank factorization.

Let $X_0 \in A\{1,2\}$ have a full-rank factorization $X_0 = YZ$. In this case, $Y \in \mathbb{C}_r^{m\times r}$, $Z \in \mathbb{C}_r^{r\times n}$, and the matrix equation (2) yields

$$YZAYZ = YZ.$$

Multiplication of the last identity on the left by $Y^{(1)}$ and on the right by $Z^{(1)}$ gives $ZAY = I_r$. Let Y_s denote the first s columns of Y and let Z_s denote the first s rows of Z. Then, both Y_s and Z_s are of full rank s, and $ZAY = I_r$ implies

$$Z_sAY_s = I_s.$$

Now, let

$$X_s = Y_sZ_s.$$

Then, $\mathrm{rank}(X_s) = s$ and

$$X_sAX_s = X_s.$$

Lemma 1.3.21. *Let $A \in \mathbb{C}_r^{m\times n}$ and $0 < s \leq r$. Then*

$$\begin{aligned} A\{2\}_s &= \left\{YZ \mid\ Y \in \mathbb{C}^{n\times s},\ Z \in \mathbb{C}^{s\times m},\ ZAY = I_s\right\} \\ A\{1,2\} &= \left\{YZ \mid\ Y \in \mathbb{C}^{n\times r},\ Z \in \mathbb{C}^{r\times m},\ ZAY = I_r\right\}. \end{aligned} \tag{1.37}$$

Definition and basic properties of the $A^{(2)}_{T,S}$-inverse

Definition 1.3.6. *Let $A \in \mathbb{C}^{m\times n}_r$, T is a subspace of $\mathbb{C}^n$ of dimension $t \le r$ and S is a subspace of $\mathbb{C}^m$ of dimension $m-t$, then A has a $\{2\}$-inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$, in which case X is unique and it is denoted by $A^{(2)}_{T,S}$.*

Lemma 1.3.22. *Let $A \in \mathbb{C}^{m\times n}$ be an arbitrary matrix, T is a subspace of $\mathbb{C}^n$ and S is a subspace of $\mathbb{C}^m$ such that $A(T) \oplus S = \mathbb{C}^m$. Then the matrix A can be written in the following way:*

$$A \sim \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A^{(2)}_{T,S}A) \end{bmatrix} \to \begin{bmatrix} AT \\ S \end{bmatrix}, \tag{1.38}$$

where A_1 is invertible. Moreover,

$$A^{(2)}_{T,S} \sim \begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} : \begin{bmatrix} AT \\ S \end{bmatrix} \to \begin{bmatrix} T \\ \mathcal{N}(A^{(2)}_{T,S}A) \end{bmatrix}.$$

The outer generalized inverse with prescribed range T and null-space S is a generalized inverse of special interest in matrix theory. The reason of the importance of this inverse is the fact that: the Moore-Penrose inverse $A^\dagger$, the weighted Moore-Penrose inverse $A^\dagger_{M,N}$, the Drazin inverse A^{D}, the weighted Drazin inverse $A_{d,W}$, the group inverse $A^\#$, the Bott-Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott-Duffin inverse $A^{(+)}_{(L)}$ are all $\{2\}$-generalized inverses of A with prescribed range and null space.

Lemma 1.3.23. *Let $A \in \mathbb{C}^{m\times n}_r$ and $p = \operatorname{ind}(A)$. Then the following representations are valid:*

(a) $A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$,

(b) $A^\dagger_{M,N} = A^{(2)}_{\mathcal{R}(N^{-1}A^*M),\mathcal{N}(N^{-1}A^*M)}$.

Also, the following statements hold in the case $m = n$:

(c) $A^{\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$, $k = \operatorname{ind}(A)$,

(d) $A^\# = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}$ *if and only if* $k = \operatorname{ind}(A) = 1$,

(e) $A_{d,W} = A^{(2)}_{\mathcal{R}(A(WA)^k),\mathcal{N}(A(WA)^k)}$, $k = \operatorname{ind}(A)$.

The Urquhart formula was originated [150] and later extended in [156, Theorem 1.3.3] and [8, Theorem 13, P. 72]. We restate it in Lemma 5.5.1.

Lemma 1.3.24. Urquhart formula.
Let $A \in \mathbb{C}^{m\times n}_r$, $U \in \mathbb{C}^{n\times p}$, $V \in \mathbb{C}^{q\times m}$ and $X = U(VAU)^{(1)}V$, where $(VAU)^{(1)}$ is a fixed but arbitrary element of $(VAU)\{1\}$. Then
(a) $X \in A\{1\}$ *if and only if* $\operatorname{rank}(VAU) = r$;
(b) $X \in A\{2\}$ *and* $\mathcal{R}(X) = \mathcal{R}(U)$ *if and only if* $\operatorname{rank}(VAU) = \operatorname{rank}(U)$;
(c) $X \in A\{2\}$ *and* $\mathcal{N}(X) = \mathcal{N}(V)$ *if and only if* $\operatorname{rank}(VAU) = \operatorname{rank}(V)$);
(d) $X = A^{(2)}_{\mathcal{R}(U),\mathcal{N}(V)}$ *if and only if* $\operatorname{rank}(VAU) = \operatorname{rank}(U) = \operatorname{rank}(V)$;
(e) $X = A^{(1,2)}_{\mathcal{R}(U),\mathcal{N}(V)}$ *if and only if* $\operatorname{rank}(VAU) = \operatorname{rank}(U) = \operatorname{rank}(V) = r$.

Using a proper combination of the Drazin inverse and the Moore-Penrose inverse, Malik and Thome [86] presented a new generalized inverse of a square matrix of an arbitrary index, which is called the DMP inverse and defined as $A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^\dagger$, for $A \in \mathbb{C}^{n\times n}$. Recall that the DMP inverse of A is the unique solution of the following equations:

$$XAX = X, \qquad XA = A^{\mathrm{D}}A, \qquad A^kX = A^kA^\dagger, \quad k = \operatorname{ind}(A).$$

Notice that the MPD inverse of A, as the dual DMP inverse, was defined as $A^{\dagger,\mathrm{D}} = A^\dagger AA^{\mathrm{D}}$ [86]. The DMP inverse for a Hilbert space operator was investigated in [106, 180] as generalizations of the DMP inverse for a square matrix.

Composing the Drazin inverse with the Moore-Penrose inverse, Malik and Thome [86] defined two new generalized inverses of a square matrix of an arbitrary index, which are known as the *DMP inverse* and *MPD inverse*. The DMP inverse of $A \in \mathbb{C}^{n\times n}$ (denoted by $A^{\mathrm{D},\dagger}$) is the unique solution to the following equations:

$$XAX = X, \qquad XA = A^{\mathrm{D}}A, \qquad A^kX = A^kA^\dagger, \quad k = \operatorname{ind}(A). \tag{1.39}$$

Recall that

$$A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^\dagger.$$

The MPD inverse of A, as the dual DMP inverse, was defined in [86] as

$$A^{\dagger,\mathrm{D}} = A^\dagger AA^{\mathrm{D}}.$$

1.4 Idempotent matrices and projectors

Idempotent matrices and projectors are very important notions and appear in numerous problems concerning various generalized inverses.

Lemma 1.4.1. *Let $E \in \mathbb{C}^{n\times n}$ be idempotent. Then E possesses the following properties:*
(a) *E^* and $I - E$ are idempotent.*
(b) *The eigenvalues of E are 0 and 1. The multiplicity of the eigenvalue 1 is equal to* $\operatorname{rank}(E)$.
(c) $\operatorname{rank}(E) = \operatorname{tr}(E)$.
(d) $E(I - E) = (I - E)E = O$.
(e) *$Ex = x$ if and only if $x \in \mathcal{R}(E)$.*
(f) $E \in E\{1, 2\}$.
(g) $\mathcal{N}(E) = \mathcal{R}(I - E)$.

The transformation denoted by $P_{L,M}$ carries any $x \in \mathbb{C}^n$ into its projection on L along M. The transformation $P_{L,M}$ is called the projector on L along M, or, oblique projector. It is known that the projector is a linear transformation.

Proposition 1.4.1. *For every idempotent matrix $E \in \mathbb{C}^{n\times n}$, the subspaces $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are complementary and satisfy*

$$E = P_{\mathcal{R}(E),\mathcal{N}(E)}.$$

1.5 Least squares and best approximate solutions

The Moore-Penrose inverse and certain solutions to some of Penrose equations play fundamental role concerning solutions to the general SoLE

$$Ax = b, \quad A \in \mathbb{C}^{m\times n},\ b \in \mathbb{C}^m \tag{1.40}$$

with respect to unknowns $x \in \mathbb{C}^n$. Fundamental result is restated in Theorem 3.1.

Theorem 1.5.1. *The linear system* (3.1) *is solvable if and only if $b \in \mathcal{R}(A)$. Equivalently,* (3.1) *has a solution if and only if $AA^\dagger b = b$.*

In this case, a general solution to (3.1) *is of the form*

$$x = A^\dagger b + (I - A^\dagger A)y, \quad \textit{for arbitrary } y \in \mathbb{C}^n. \tag{1.41}$$

An arbitrary inconsistent SoLE, given by

$$Ax = b, \quad A \in \mathbb{C}^{m\times n},\ b \notin \mathcal{R}(A) \tag{1.42}$$

has no solution. Then the problem is to find an x which minimizes the residual $Ax - b$. Then a vector $u \in \mathbb{C}^n$ is called a *least squares solution* to (3.3) if

$$\|Au - b\| \le \|Ax - b\|, \quad \forall x \in \mathbb{C}^n.$$

The following proposition, restated from [8], shows that $\|Ax-b\|$ is minimized by the vector $x = A^{(1,3)}b$. This statement establishes very important relation between the set of $\{1,3\}$-inverses and the least-squares solutions of the system (3.1).

Proposition 1.5.1. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then $\|Ax - b\|$ is smallest when $x = A^{(1,3)}b$, where $A^{(1,3)} \in A\{1, 3\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, $\|Ax - b\|$ is smallest when $x = Xb$, then $X \in A\{1, 3\}$.*

Since $A^{(1,3)}$ inverse of a matrix is not unique, as a consequence, a SoLE has many least-squares solutions in general. However, among all least-squares solutions of a given SoLE, there exists only one such *solution of minimum norm*.

Definition 1.5.1. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. A vector $\hat{x}$, which satisfies the minimization problem*

$$\|\hat{x}\| = \min_{x\in\mathbb{C}^n} \|x\|, \quad \text{subject to } Ax = b, \tag{1.43}$$

is called a minimal-norm solution *of the system $Ax = b$.*

The next proposition, restated from [8], establishes a relation between $\{1,4\}$-inverses and the minimum-norm solutions of the linear system $Ax = b$.

Proposition 1.5.2. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. If $Ax = b$ is consistent, the unique solution x for which $\|x\|$ is smallest is given by $x = A^{(1,4)}b$, where $A^{(1,4)} \in A\{1,4\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ is such that, whenever $Ax = b$ has a solution, $x = Xb$ is the solution of minimal-norm, then $X \in A\{1,4\}$.*

The *least-squares solution of minimum norm* is known as *best approximate solution.* Joining the results from Proposition 3.1.1 and Proposition 3.1.2, we are coming to the most important property of the Moore-Penrose inverse.

Corollary 1.5.1. (Penrose 1955) [122] *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then, among the least-squares solutions of $Ax = b$, the solution $A^\dagger b$ is the one of minimum-norm. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that Xb is the* minimal-norm least-squares solution *of $Ax = b$ for all b, then $X = A^\dagger$.*

In the essence, Corollary 3.1.1 shows that $A^\dagger b$ is the minimal-norm least-squares solution of the linear system $Ax = b$. This fact caused a dramatic increase of the interest in the generalized inverses theory.

Furthermore, the next proposition characterizes the set of all least-squares solutions of a given SoLE.

Proposition 1.5.3. (Nashed 1970, 1976) [119, 118] For $A \in \mathbb{C}^{m\times n}$, the set S of all least-squares solutions of the system $Ax = b$ is given by

$$S = A^\dagger b \oplus \mathcal{N}(A) = \{A^\dagger b + (I - A^\dagger A)y| \ \ y \in \mathbb{C}^n\}.$$

These results are extended in solving the linear matrix equations (LME) $AX = B$. More precisely, the Moore-Penrose inverse satisfies the following inequalities [122]:

$$\|AX - B\| \geq \|AA^\dagger B - B\| \tag{1.44}$$

for all X, with equality in (3.5) if and only if

$$X = A^\dagger B + (I - A^\dagger A)L,$$

where L is arbitrary matrix of appropriate dimensions. Moreover,

$$\|A^\dagger B + (I - A^\dagger A)L\| \geq \|A^\dagger B\|, \tag{1.45}$$

with equality in (3.6) if and only if $(I - A^\dagger A)L = 0$.

Penrose's inequalities (3.5) and (3.6) has been extended in [85] to the supremum norm and the L_p norm as well as to the set of $\{1,3\}$ inverses. This result is restated here for complex matrices.

Proposition 1.5.4. *Let $A \in \mathbb{C}^{m\times n}$ and $A^{(1,3)}$ be an $\{1,3\}$ inverse of A. Then for all X*

$$\|AX - B\| \geq \|AA^{(1,3)}B - B\|, \tag{1.46}$$

with equality in (3.7) *if and only if*

$$X = A^{(1,3)}B + (I - A^{(1,3)}A)L,$$

where L is arbitrary. Furthermore, the choice $A^{(1,3)} := A^\dagger$ leads to the least squares solution of minimum norm, equal to $A^\dagger B$:

$$\|A^\dagger B + (I - A^\dagger A)L\| \geq \|A^\dagger B\|. \tag{1.47}$$

1.6 Minimal properties of generalized inverses

For $A \in \mathbb{C}^{n\times n}$, there exists the Drazin inverse of A (denoted by A^{D}), i.e., the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$(1^k) \quad A^{k+1}X = A^k, \qquad (2) \quad XAX = X, \qquad (5) \quad AX = XA,$$

where $k = \mathrm{ind}(A) = \min\{k \geq 0| \ \mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})\}$ is the index of A. In the case $\mathrm{ind}(A) = 1$, the Drazin inverse A^{D} reduces to the group inverse of A (denoted by $A^{\#}$). For various applications of the Drazin inverse see [8, 12].

An outer inverse (or $\{2\}$-inverse) of $A \in \mathbb{C}^{m\times n}$ is a matrix $X \in \mathbb{C}^{n\times m}$ which satisfies the equation $XAX = X$. The outer inverses of A with determined null space and range attracted attention of many authors because of their uniqueness and generality.

Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. The outer inverse of A with the range T and the null-space S (denoted by $A^{(2)}_{T,S}$) is a matrix $X \in \mathbb{C}^{n\times m}$ such that

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S,$$

Recall that A has an outer inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$, and in this case $X = A^{(2)}_{T,S}$ is unique [8, 156].

The Moore-Penrose inverse $A^\dagger$, the Drazin inverse A^{D} and the group inverse $A^{\#}$ can be presented as particular generalized inverses $A^{(2)}_{T,S}$ for appropriate choice of the matrix G such that $T = \mathcal{R}(G)$ and $S = \mathcal{N}(G)$. For example, the next statement is valid for arbitrary matrix A:

$$A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}. \tag{1.48}$$

The next identities (see [8, 156]) are satisfied for arbitrary $A \in \mathbb{C}^{n\times n}$:

$$A^{\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \quad k = \mathrm{ind}(A); \quad A^{\#} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}, \quad 1 = \mathrm{ind}(A). \tag{1.49}$$

1.6.1 Least-square properties of the Drazin-inverse solution

In the papers [12, 164, 171], the authors present some minimal properties of the Drazin-inverse solution. It can be argued that, in some way, these properties correspond to the properties of the Moore-Penrose inverse solution. Namely, in [12] it is shown that if $b \in \mathcal{R}(A^k)$, where $k = \mathrm{ind}(A)$, then the Drazin-inverse solution is the unique solution of the system $Ax = b$ which belongs to $\mathcal{R}(A^k)$. Also, Wei et al. in [164, 171] proved that the Drazin-inverse solution of the system $Ax = b$ is a solution of minimum P-norm, where P is the matrix included in the Jordan decomposition $A = PJP^{-1}$ of the matrix A.

The obtained results related to the Drazin-inverse solution of a given system SoLE, are inspiration to investigate possibilities if they can be used in order to calculate the Drazin inverse of a given matrix, i.e., to find the Drazin-inverse solution of the matrix equation $AXB = D$, in general. With appropriate modifications, it is possible to find the solution in the form $A^{\mathrm{D}}GB^{\mathrm{D}}$. The matrix $A^{\mathrm{D}}GB^{\mathrm{D}}$ is not always a solution of the matrix equation $AXB = D$, but however it can be always used in order to calculate the Drazin inverse of arbitrary matrix.

The results of this section are complement to the results investigated in [171]. Namely, they are motivated form the idea of defining a gradient iterative method for computing the Drazin-inverse solution of the system (3.9). The goal is achieved by establishing a relation between the Drazin-inverse solution and the linear system (3.9).

Theorem 1.6.1. [166] *Each solution to*

$$Ax = b, \quad b \in \mathcal{R}(A^k),\ k = \mathrm{ind}(A) \tag{1.50}$$

is also a solution to

$$A^{p+1}x = A^p b \qquad p \ge k, \tag{1.51}$$

but the opposite statement does not hold.

Proof. Clearly $Ax = b$, $b \in \mathcal{R}(A^k)$ implies $A^p(Ax - b) = 0$ for $p \ge k$.

On the other hand, Wei in [164] proved that the general solution of (3.9) is given by

$$x = A^{\mathrm{D}}b + A^{k-1}(I - A^{\mathrm{D}}A)z, \tag{1.52}$$

where z is an arbitrary vector. □

The solution $A^{\mathrm{D}}b$ is known as the Drazin-inverse solution of (3.9).

Remark that the opposite statement is not valid, since not every element from $\mathcal{N}(A^k)$ can be represented as $A^{k-1}(I - A^{\mathrm{D}}A)z$, $z \in \mathbb{C}^n$ is arbitrary. Consequently, not every solution of (3.10) is a solution of the equation (3.9) nor a solution of the equation (3.1).

Theorem 1.6.2. [171] *Consider $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{ind}(A)$. The Drazin inverse solution $A^{\mathrm{D}}b$ is the unique solution in $\mathcal{R}(A^k)$ to the system*

$$A^{k+1}x = A^k b. \tag{1.53}$$

Theorem 1.6.3. [171] *Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ and $k = \mathrm{ind}(A)$. The set of all solutions of the equation* (3.12) *is given by*

$$x = A^{\mathrm{D}}b + \mathcal{N}(A^k). \tag{1.54}$$

1.6.2 Least-square properties of outer inverses

The outer generalized inverses with prescribed range and null-space are very important in matrix theory. The $\{2\}$-inverses have application in defining iterative methods for solving the nonlinear equations [8], in statistics [45] as well as in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverses [118].

Outer inverses with prescribed range and null space are useful in solving the restricted SoLE. This application is based on the following essential result from [22]:

Proposition 1.6.1. [22] *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$, and let the condition*

$$b \in AT, \quad \dim(AT) = \dim(T)$$

be satisfied. Then the unique solution to the constrained SoLE

$$Ax = b, \quad x \in T$$

is given by

$$x = A^{(2)}_{T,S}b,$$

for any subspace S of $\mathbb{C}^m$ satisfying $AT \oplus S = \mathbb{C}^m$.

Further investigations show that some new classes of generalized inverses are applicable in solving corresponding unconstrained and constrained SoLE. These generalized inverses are composed of appropriate outer inverses and the Moore-Penrose inverse and surveyed in the subsequent Section 2.3.

1.7 Nonlinear optimization and generalized inversion

It is known that calculation of the inverse matrix can be included in finding solutions to some optimization models. On the other hand, the calculation of the inverse matrix can be defined on the basis of certain optimization models. In his famous paper [122], Penrose was the first who showed the close connection between the Moore-Penrose inverse and the least-squares solution problem of a system of linear equations. The same principle is extended to solving matrix equations. Since the least-squares problem represents a special case of the nonlinear optimization problems, a close relationship between the theory of generalized inverses and the optimization theory is evident. Additionally, the discovered minimal properties of the solution of a linear system of equations, obtained with the usage of the Moore-Penrose inverse, brought to intensive usage of the optimization methods in numerical computation of generalized inverses.

The theory of optimization represents a very important mathematical discipline and finds a great application, not only in the theory of applied mathematics, but also in many practical disciplines such as: production, aviation, management, sociology, genetic etc. Moreover, the process of evolution follows the principles of optimization. Although the optimization theory is a part of everyday life for a very long time, this science has faced an important development in the last five decades. The subject is involved in the process of finding optimal solution of problems which are defined mathematically. More precisely, given a practical problem, the "best" solution to the problem can be found from lots of schemes by means of scientific methods and tools. It involves the study of optimality conditions of the problems, the construction of model problems, the determination of algorithmic method of solution, the establishment of convergence theory of the algorithms, and numerical experiments with typical problems and real life problems.

Chapter 2

Composite generalized inverses

To extend and unify the notions of the core, dual core, DMP, MPD, CMP, MPCEP and $*$CEPMP inverses, the OMP, MPO and MPOMP inverses were introduced in [108] composing an arbitrary outer inverse and the Moore-Penrose inverse. We use *composite outer inverses* to denote one common term for all appearances of the OMP, MPO and MPOMP inverses. The core inverse, the DMP inverse and the $*$CEPMP inverse are particular cases of the OMP inverse, because the group, Drazin and dual core-EP inverses are particular outer inverses. The MPD, dual core and MPCEP inverses are special cases of MPO inverses. Also, the MPOMP inverses include the CMP and Moore-Penorse inverses.

Different characterizations and representations of composite outer inverses are surveyed and analysed in this chapter based on [136]. New characterizations and representations of the core, dual core, DMP, MPD, CMP, MPCEP and $*$CEPMP inverses are obtained in [136] as particular consequences of these results.

2.1 Survey of composite outer inverses involving the Moore-Penrose inverse

Proper combinations of particular outer inverses with the MP inverse is a popular trend in research of generalized inverses. Main properties, representations and characterizations of these inverses have been investigated in a number of papers. Main interest will be possible solutions of restricted systems of linear equations in terms of various composite outer inverses.

2.1.1 Core-EP inverse

To solve some types of matrix equations, Baksalary and Trenkler introduced the core inverse of a square matrix in [2]. Observe that a matrix $A \in \mathbb{C}^{n\times n}$ has the core inverse if and only if $\mathrm{ind}(A) \leq 1$. In this case, the core inverse of A can be expressed by $A^{\#\!\!\!\!\bigcirc} = A^{\#}AA^{\dagger}$. Kurata [71] established the maximal classes of matrices Q and S included in $A\{1\}$, for which QAS coincides with the core inverse of A.

The core-EP inverse was presented by Prasad and Mohana in [123] for a square matrix, which is not essentially of index one, generalizing the notion of the core inverse.

Definition 2.1.1. [123] *Let $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$. A matrix $X \in \mathbb{C}^{n\times n}$, denoted by $A^{\dagger\!\!\!\!\bigcirc}$, is called the core-EP inverse of A if it satisfies*

$$XAX = X, \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k).$$

Wang in [151] introduced a new decomposition for the core-EP inverse which arises from the Schur decomposition.

Lemma 2.1.1. [151] *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and $\mathrm{rank}(A^k) = r$. The Schur form of A is given by*

$$A = U \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} U^*, \tag{2.1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, T_1 is a $r \times r$ nonsingular upper-triangular matrix and T_3 is a nilpotent matrix with index k. The core-EP decomposition of A is defined in [151] as the sum

$A = A_1 + A_2$, *where* $A_1 \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A_1) = 1$, $A_2^k = 0$, $A_1^* A_2 = A_2 A_1^* = 0$. *One possible decomposition satisfying these conditions was originated in* [151], *as follows:*

$$A_1 = U \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} U^* = A^k (A^k)^\dagger A,$$
$$A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & T_3 \end{bmatrix} U^* = A - A^k (A^k)^\dagger A.$$

Then the core-EP inverse can be expressed in the form [151]

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = A^{\mathrm{D}} A^k (A^k)^\dagger = A^k (A^{k+1})^\dagger.$$

Also, the following representation from [123] is known:

$$A^{\textcircled{\dagger}} = A^k \left(\left(A^k\right)^* A^{k+1} \right)^- \left(A^k\right)^* = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}\left((A^k)^*\right)}. \tag{2.2}$$

In [42] the core-EP inverse is represented as the combination of the Drazin inverse, matrix power and the Moore-Penrose inverse:

$$A^{\textcircled{\dagger}} := A^{\mathrm{D}} A^k (A^k)^\dagger.$$

If $\mathrm{ind}(A) = 1$, then $A^{\textcircled{\dagger}}$ becomes his predecessor, known as the core inverse of A and denoted by $A^{\textcircled{\#}}$ [2]. Thus, $A^{\textcircled{\#}} = A^\# A A^\dagger$.

The dual core-EP inverse is given by $A_{\textcircled{\dagger}} = (A^k)^\dagger A^k A^{\mathrm{D}}$. As a consequence, the dual core inverse of A (denoted by $A_{\textcircled{\#}}$) is expressed as $A_{\textcircled{\#}} = A^\dagger A A^\#$.

In the recent years, the core-EP inverse has become a very popular kind of outer inverse. Various representations of the core-EP inverse were investigated in [81, 151, 185]. The core-EP inverse was extended to rectangular matrices in [40], to bounded linear Hilbert space operators in [106, 104], to elements of rings in [42], while the extensions to tensors and quaternion matrices are originated in [128] and [62], respectively.

2.1.2 DMP and MPD inverse

Two new generalized inverses of a square matrices of arbitrary index are defined in [86] as a hybrid combination of the Drazin inverse with the Moore-Penrose inverse. These generalized inverses are known as the DMP inverse and MPD inverse. The DMP inverse of $A \in \mathbb{C}^{n\times n}$ (denoted by $A^{\mathrm{D},\dagger}$) is the unique solution to the following matrix equations:

$$XAX = X, \qquad XA = A^{\mathrm{D}} A, \qquad A^k X = A^k A^\dagger,\ k = \mathrm{ind}(A). \tag{2.3}$$

The system of equations (2.3) is consistent and has a unique solution $A^{\mathrm{D}} A A^\dagger$ [86, Theorem 2.2], denoted by $A^{\mathrm{D},\dagger} := A^{\mathrm{D}} A A^\dagger$ and termed as the DMP inverse of A. Deng and Yu, [180], Liu and Cai [75] described the range space and null space of the DMP inverse, as follows:

$$\mathcal{R}(A^{\mathrm{D},\dagger}) = \mathcal{R}(A^k), \qquad \mathcal{N}(A^{\mathrm{D},\dagger}) = \mathcal{N}(A^k A^\dagger),\ k = \mathrm{ind}(A).$$

The DMP inverse is just the outer inverse satisfying $A^{\mathrm{D},\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}\left(A^k A^\dagger\right)}$ [29, 180]. Main results concerning the DMP inverse were proved in [62, 75, 79, 89, 106, 154, 180]. Recently, Meng in [89] extended the definition of the DMP inverse to rectangular matrices. Zhu in [**?**] introduced the pseudo DMP inverse in a ring as an extension of the DMP inverse. In [153], the authors developed an algorithm for computing the DMP inverse on the basis of the Cayley-Hamilton theorem. Various representations for the DMP inverse can be found in [79]. Iterative method for finding DMP inverse was proved in [75]. The DMP inverse was generalized to operators in [106, 180] and to tensors in [154].

The MPD inverse of A is defined as the dual to the DMP inverse and defined by $A^{\dagger,\mathrm{D}} := A^\dagger A A^{\mathrm{D}}$ [86].

In [153], the authors developed an algorithm for computing the DMP inverse on the basis of the Cayley-Hamilton theorem. Various representations for the DMP inverse can be found in [39, 62, 79]. Iterative method for finding DMP inverse was proved in [75]. Meng in [89] extended the definition of the DMP inverse to rectangular matrices. Zhu in [186] introduced the pseudo DMP inverse in a ring as an extension of the DMP inverse. The DMP inverse was generalized to operators in [101, 106, 180] and to tensors in [154]. Applications of the DMP and MPD inverses in solving some restricted quaternion matrix equations can be found in [64, 70].

2.2 Overview of remaining composed generalized inverses

The *CMP inverse* is a new generalized inverse of a square matrix A, which was originated by Mehdipour and Salemi [88] in terms of the core part $AA^{\mathrm{D}}A$ of A and $A^{\dagger}$. The CMP inverse of $A \in \mathbb{C}^{n\times n}$ (denoted by $A^{c,\dagger}$) is the unique solution of the next matrix equations:

$$XAX = X, \quad AXA = AA^{\mathrm{D}}A, \quad AX = AA^{\mathrm{D}}AA^{\dagger}, \quad XA = A^{\dagger}AA^{\mathrm{D}}A.$$

It is known that

$$A^{c,\dagger} = A^{\dagger}AA^{\mathrm{D}}AA^{\dagger}.$$

Various representations and main properties of the CMP inverse can be found in [39, 96, 97, 100, 103, 110, 154, 177]. Maximal classes of matrices determining the DMP and CMP inverses were developed in [39].

Let $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$. There exists the *core-EP inverse* (or CEP inverse) of A (denoted by $A^{\textcircled{\dagger}}$) as the unique matrix $X \in \mathbb{C}^{n\times n}$ such that [123]:

$$X = XAX, \quad \mathcal{R}(A^k) = \mathcal{R}(X) = \mathcal{R}(X^*).$$

The core-EP inverse is defined by the matrix expression defined by the Drazin inverse, rank-invariant matrix power and its pseudoinverse [42]:

$$A^{\textcircled{\dagger}} = A^{\mathrm{D}}A^k(A^k)^{\dagger}.$$

The *dual core-EP inverse* (or $*$core-EP) inverse of A is uniquely determined matrix $X \in \mathbb{C}^{n\times n}$ (denoted by $A_{\textcircled{\dagger}}$) such that

$$X = XAX, \quad \mathcal{R}\left((A^k)^*\right) = \mathcal{R}(X) = \mathcal{R}(X^*).$$

The dual core-EP is defined inverting the order of terms included in the core-EP inverse:

$$A_{\textcircled{\dagger}} = (A^k)^{\dagger}A^kA^{\mathrm{D}},$$

which is the unique solution to

$$XAX = X, \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}((A^k)^*).$$

If $\mathrm{ind}(A) = 1$, then $A^{\textcircled{\dagger}}$ reduces to the *core inverse* of A (denoted by $A^{\textcircled{\#}}$) [2]. It is known that

$$A^{\textcircled{\#}} = A^{\#}AA^{\dagger}.$$

The *dual core* (or $*$core) inverse of A (denoted by $A_{\textcircled{\#}}$) is expressed as

$$A_{\textcircled{\#}} = A^{\dagger}AA^{\#}.$$

Recently, many authors have studied the core and core-EP inverse for matrices. Bordering technique and iterations for finding the core-EP inverse were given in [124, 125]. Several characterizations and representations for core-EP inverse were proposed in [43, 81, 82, 105, 102, 158, 185, 183, 184]. Further, for the core-EP inverse, some limit representations were studied in [185]. Continuity of the core-EP inverse was considered in [43]. The core-EP was generalized to rectangular matrices using a weight matrix in [40], to bounded linear Hilbert space operators in [104, 106], to Banach algebra elements in [93], to elements in rings [42, 94, 187], to quaternion matrices in [61] and to tensors in [128].

The weak group inverse was introduced in [152] by the expression

$$A^{\textcircled{w}} := (A^{\textcircled{\dagger}})^2A,$$

where $A \in \mathbb{C}^{n\times n}$. The latest results related to the weak group inverse are included in [41, 117]. Representations suitable for numerical calculation of the *weak group inverse* were introduced in [116] for $l \geq k = \mathrm{ind}(A)$:

$$A^{\textcircled{w}} = \left(A^{\mathrm{D}}\right)^2 A^l(A^l)^{\dagger}A = A^l\left(A^{l+2}\right)^{\dagger}A.$$

Applying the Moore-Penrose inverse and the core-EP inverse, the *MPCEP inverse* for a Hilbert space operator was presented in [19]. The MPCEP inverse of $A \in \mathbb{C}^{n\times n}$ (denoted by $A^{\dagger,\textcircled{\dagger}}$) is the unique solution to the system of matrix equations:

$$XAX = X, \qquad AX = AA^{\textcircled{\dagger}}, \qquad XA = A^{\dagger}AA^{\textcircled{\dagger}}A.$$

Also, by [19], we have
$$A^{\dagger,\textcircled{\dagger}} = A^{\dagger}AA^{\textcircled{\dagger}}.$$
The $*$*CEPMP inverse* is presented in [19] as
$$A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}}AA^{\dagger}$$
and it is the unique solution to the system of matrix equations
$$XAX = X, \qquad AX = AA_{\textcircled{\dagger}}AA^{\dagger}, \qquad XA = A_{\textcircled{\dagger}}A.$$
Some representations and properties of the MPCEP inverse were introduced in [107, 64, 66, 148]. The MPCEP and $*$CEPMP inverses are applied to solve quaternion matrix equations with constrains in [65, 63, 69].

The W-weighted Drazin inverse was introduced in [25] as an extension of the Drazin inverse to rectangular matrices. In the case that $W \in \mathbb{C}^{n\times m}$ and $A \in \mathbb{C}^{m\times n}$, the W-weighted Drazin inverse $A^{\mathrm{D},W}$ of A can be considered as
$$A^{\mathrm{D},W} = \left[(AW)^{\mathrm{D}}\right]^2 A = A\left[(WA)^{\mathrm{D}}\right]^2.$$
More details about A-weighted Drazin inverse can be found in [156].

The core-EP inverse was generalized to rectangular matrices in [40]. For $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$, the unique matrix $A^{\textcircled{\dagger},W} = X \in \mathbb{C}^{m\times n}$ is the W-weighted core-EP inverse of A if
$$WAWX = (WA)^k[(WA)^k]^{\dagger}, \quad \mathcal{R}((AW)^k) = \mathcal{R}(X).$$
Recall that, by [104], $A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2$. For more recent results considering W-weighted core-EP inverse see [39, 44, 78]. The W-weighted core-EP inverse was studied for bounded linear Hilbert space operators in [97, 99, 104], for elements of C^*-algebras in [105] and for elements of rings in [187].

2.3 OMP inverses

Composing the outer inverse and the Moore-Penrose inverse in adequate ways, three outer inverses of a rectangular matrix were presented in [108]. In this section, we consider main properties of the first of them, which is called the OMP inverse.

2.3.1 Characterizations of the OMP inverse

We characterize the OMP inverse from algebraic and geometrical approaches in this subsection.

The OMP inverse was introduced as a solution to some type of matrix equations in [108, Theorem 2.1].

Theorem 2.3.1. [108, Theorem 2.1] *If $A \in \mathbb{C}^{m\times n}_{T,S}$, the system of matrix equations*
$$XAX = X, \quad AX = AA^{(2)}_{T,S}AA^{\dagger}, \quad XA = A^{(2)}_{T,S}A \tag{2.4}$$
is consistent and its unique solution is $X := A^{(2)}_{T,S}AA^{\dagger}$.

Proof. If $X := A^{(2)}_{T,S}AA^{\dagger}$, then
$$\begin{aligned} AX &= AA^{(2)}_{T,S}AA^{\dagger}, \\ XA &= A^{(2)}_{T,S}AA^{\dagger}A = A^{(2)}_{T,S}A, \\ XAX &= A^{(2)}_{T,S}AX = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^{\dagger} = A^{(2)}_{T,S}AA^{\dagger} = X. \end{aligned}$$
Therefore, $X = A^{(2)}_{T,S}AA^{\dagger}$ satisfies the system of equations (2.4).

Assume that two matrices X and X_1 satisfy (2.4), that is, (2.4) holds, $X_1AX_1 = X_1$, $AX_1 = AA^{(2)}_{T,S}AA^{\dagger}$ and $X_1A = A^{(2)}_{T,S}A$. Hence,
$$X = (XA)X = (A^{(2)}_{T,S}A)X = X_1(AX) = X_1(AA^{(2)}_{T,S}AA^{\dagger}) = X_1AX_1 = X_1$$
and so the system (2.4) has the unique solution. □

Definition 2.3.1. [108] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The* OMP *(or outer Moore-Penrose) inverse of A is defined as*
$$A^{(2),\dagger}_{T,S} = A^{(2)}_{T,S}AA^{\dagger}.$$

Significant special cases of OMP inverses are given below:

(i) When $m = n$, $\operatorname{ind}(A) = k$ and $A^{(2)}_{T,S} := A^{\mathrm{D}}$, the system (2.4) becomes

$$XAX = X, \qquad AX = AA^{\mathrm{D}}AA^{\dagger} \qquad \text{and} \qquad XA = A^{\mathrm{D}}A. \tag{2.5}$$

According to Theorem 2.3.1, the matrix $X := A^{\mathrm{D}}AA^{\dagger} = A^{\mathrm{D},\dagger}$ is the unique solution to (2.5). So, the OMP inverse reduces to the DMP inverse in the case $A^{(2)}_{T,S} := A^{\mathrm{D}}$. In view of basic properties of the Drazin inverse, notice that (2.5) implies (1.39).

(ii) If $m = n$, $\operatorname{ind}(A) = 1$ and $A^{(2)}_{T,S} := A^{\#}$, the system (2.4) is converted into the system

$$XAX = X, \quad AX = AA^{\dagger}, \quad XA = A^{\#}A,$$

which has the unique solution $X = A^{\#}AA^{\dagger} = A^{\textcircled{\#}}$ by Theorem 2.3.1. Thus, the OMP inverse becomes the core inverse in this case.

(iii) For $m = n$, $\operatorname{ind}(A) = k$ and $A^{(2)}_{T,S} := A_{\textcircled{\dagger}}$, the $*$CEPMP inverse $A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}}AA^{\dagger}$ is the unique solution to the matrix system [19]

$$XAX = X, \quad AX = AA_{\textcircled{\dagger}}AA^{\dagger}, \quad XA = A_{\textcircled{\dagger}}A.$$

More characterizations of the OMP inverse arising from algebraic approach were proved in [108, Theorem 2.2].

Theorem 2.3.2. [108, Theorem 2.2] *If $A \in \mathbb{C}^{m\times n}_{T,S}$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times m}$ *is the* OMP *inverse* $A^{(2)}_{T,S}AA^{\dagger}$ *of* A*;*

(ii) $XAX=X, \quad AXA=AA^{(2)}_{T,S}A, \quad AX=AA^{(2)}_{T,S}AA^{\dagger}, \quad XA=A^{(2)}_{T,S}A;$

(iii) $A^{(2)}_{T,S}AX = X, \quad AX = AA^{(2)}_{T,S}AA^{\dagger};$

(iv) $XAA^{(2)}_{T,S}AA^{\dagger} = X, \quad XA = A^{(2)}_{T,S}A;$

(v) $XAA^{(2)}_{T,S}AX=X, \quad AA^{(2)}_{T,S}AX=AA^{(2)}_{T,S}AA^{\dagger}, \quad XAA^{(2)}_{T,S}A=A^{(2)}_{T,S}A;$

(vi) $XAA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}A = AA^{(2)}_{T,S}A,$
$AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}AA^{\dagger}, \quad XAA^{(2)}_{T,S}A = A^{(2)}_{T,S}A;$

(vii) $A^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}AA^{\dagger};$

(viii) $XAA^{(2)}_{T,S}AA^{\dagger} = X, \quad XAA^{(2)}_{T,S}A = A^{(2)}_{T,S}A;$

(ix) $XAA^{\dagger} = X, \quad XAA^{*} = A^{(2)}_{T,S}AA^{*};$

(x) $XAA^{\dagger} = X, \quad XA = A^{(2)}_{T,S}A.$

Proof. (i) $\Rightarrow$ (ii): The hypothesis $X = A^{(2)}_{T,S}AA^{\dagger}$ yields $AXA = AA^{(2)}_{T,S}AA^{\dagger}A = AA^{(2)}_{T,S}A$. The rest three equalities are evident by Theorem 2.3.1.

(ii) $\Rightarrow$ (iii): The assumptions $XAX = X$ and $XA = A^{(2)}_{T,S}A$ imply $A^{(2)}_{T,S}AX = XAX = X$.

(iii) $\Rightarrow$ (i): We have $X = A^{(2)}_{T,S}(AX) = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^{\dagger} = A^{(2)}_{T,S}AA^{\dagger}$.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (i): It can be verified as (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

In an analogy manner, we show that (i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (viii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ix): Using $X = A^{(2)}_{T,S}AA^{\dagger}$ and $A^{*} = A^{\dagger}AA^{*}$, we get $XAA^{\dagger} = A^{(2)}_{T,S}AA^{\dagger}AA^{\dagger} = A^{(2)}_{T,S}AA^{\dagger} = X$ and $XAA^{*} = A^{(2)}_{T,S}AA^{\dagger}AA^{*} = A^{(2)}_{T,S}AA^{*}$.

(ix) $\Rightarrow$ (x): From the condition $XAA^{*} = A^{(2)}_{T,S}AA^{*}$, we observe that

$$XA = XAA^{\dagger}A = (XAA^{*})(A^{\dagger})^{*} = A^{(2)}_{T,S}AA^{*}(A^{\dagger})^{*} = A^{(2)}_{T,S}A.$$

(x) $\Rightarrow$ (i): Notice that $XAA^{\dagger} = X$ and $XA = A^{(2)}_{T,S}A$ give $X = (XA)A^{\dagger} = A^{(2)}_{T,S}AA^{\dagger}$. $\square$

Taking, for $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$, $T := \mathcal{R}(B)$ and $S := \mathcal{N}(C)$, certain equivalent conditions for a rectangular matrix to be the OMP inverse $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger}$ were verified in [108, Theorem 2.3].

Theorem 2.3.3. [108, Theorem 2.3] *For arbitrary $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times m}$ *is the* OMP *inverse* $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger}$ *of* A*;*

(ii) $CAX = CAA^\dagger, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = X$;

(iii) $CAXA = CA, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X$;

(iv) $CAXAA^* = CAA^*, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X$;

(v) $XAB = B, \quad XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$;

(vi) $AXAB = AB, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$;

(vii) $A^*AXAB = A^*AB, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$.

Proof. Since $R(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}) = \mathcal{R}(B)$, we observe that $A^{(2)}_{R(B),N(C)}AB = B$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, for $B^{(1)} \in B\{1\}$. Also, by $\mathcal{N}(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}) = \mathcal{N}(C)$, we have $CAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = C$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}C$, for $C^{(1)} \in C\{1\}$.

(i) $\Rightarrow$ (ii): Because $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$, then $CAX = (CAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})AA^\dagger = CAA^\dagger$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$.

(ii) $\Rightarrow$ (i): From $CAX = CAA^\dagger$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = X$, it follows

$$\begin{aligned} X &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}(CAX) \\ &= (A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}C)AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger. \end{aligned}$$

(i) $\Rightarrow$ (iii): Notice that $X = A^{(2)}_{R(B),N(C)}AA^\dagger$ implies $CAXA = CAA^\dagger A = CA$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = XAA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$.

(iii) $\Rightarrow$ (iv): This implication is clear.

(iv) $\Rightarrow$ (i): Multiplying $CAXAA^* = CAA^*$ by $(A^\dagger)^*$ from the right hand side, we get $CAXA = CA$. Therefore,

$$\begin{aligned} A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXA &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}(CAXA) \\ &= \left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}C\right)A = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A \end{aligned}$$

in conjunction with the assumption $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X$ give

$$X = \left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXA\right)A^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger.$$

(i) $\Rightarrow$ (v): Applying $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AB = B$, we obtain

$$XAB = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(AA^\dagger A)B = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AB = B$$

and

$$XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X.$$

(v) $\Rightarrow$ (i): By $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ and the hypothesis $XAB = B$, we have

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = XA\left(BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right) = XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}.$$

Now, from $XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$, it follows that $X = (XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$.

(i) $\Rightarrow$ (vi): The equalities $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AB = B$ yield from $AXAB = AB$ and

$$\begin{aligned} A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger \\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X. \end{aligned}$$

(vi) $\Rightarrow$ (vii): It is evident.

(vii) $\Rightarrow$ (i): Multiplying $A^*AXAB = A^*AB$ by $(A^\dagger)^*$ from the left hand side, we obtain $AXAB = AB$. If we multiply the previous equality by $B^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ from the right hand side, then $AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. Hence,

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}) = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$$

and, by $A^{(2)}_{T,S}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X$,

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = X,$$

which completes the proof. $\square$

Notice that the core inverse and the DMP inverse are special cases of the OMP inverse. Corresponding characterizations of the DMP and core inverses are derived as consequences. Applying Theorem 2.3.2 and basic properties of the Drazin inverse, we firstly obtain characterizations of the DMP inverse.

Corollary 2.3.1. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times n}$ *is the* DMP *inverse* $A^{\mathrm{D}}AA^{\dagger}$ *of* A*;*

(ii) $XAX = X, \quad AXA = AA^{\mathrm{D}}A, \quad AX = AA^{\mathrm{D}}AA^{\dagger}, \quad XA = A^{\mathrm{D}}A;$

(iii) $A^{\mathrm{D}}AX = X, \quad AX = AA^{\mathrm{D}}AA^{\dagger};$

(iv) $XAA^{\mathrm{D}}AA^{\dagger} = X, \quad XA = A^{\mathrm{D}}A;$

(v) $XAA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AX = AA^{\mathrm{D}}AA^{\dagger}, \quad XAA^{\mathrm{D}}A = A^{\mathrm{D}}A;$

(v') $XAA^{\mathrm{D}}AX = X, \quad A^kX = A^kA^{\dagger}, \quad XA^{k+1} = A^k;$

(vi) $XAA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AXAA^{\mathrm{D}}A = AA^{\mathrm{D}}A,$
$AA^{\mathrm{D}}AX = AA^{\mathrm{D}}AA^{\dagger}, \quad XAA^{\mathrm{D}}A = A^{\mathrm{D}}A;$

(vi') $XAA^{\mathrm{D}}AX = X, \quad A^kXA^k = A^{2k-1}, \quad A^kX = A^kA^{\dagger}, \quad XA^{k+1} = A^k;$

(vii) $A^{\mathrm{D}}AX = X$ *and* $AA^{\mathrm{D}}AX = AA^{\mathrm{D}}AA^{\dagger};$

(vii') $A^{\mathrm{D}}AX = X, \quad A^kX = A^kA^{\dagger};$

(viii) $XAA^{\mathrm{D}}AA^{\dagger} = X, \quad XAA^{\mathrm{D}}A = A^{\mathrm{D}}A;$

(viii') $XAA^{\mathrm{D}}AA^{\dagger} = X, \quad XA^{k+1} = A^k;$

(ix) $XAA^{\dagger} = X, \quad XAA^* = A^{\mathrm{D}}AA^*;$

(x) $XAA^{\dagger} = X, \quad XA = A^{\mathrm{D}}A.$

In a particular case that $\mathrm{ind}(A) = 1$ in Corollary 2.3.1, we get necessary and sufficient conditions which characterize the core inverse of a square matrix.

Corollary 2.3.2. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times n}$ *is the core inverse* $A^{\#}AA^{\dagger}$ *of* A*;*

(ii) $XAX = X, \quad AXA = A, \quad AX = AA^{\dagger}, \quad XA = A^{\#}A;$

(iii) $A^{\#}AX = X, \quad AX = AA^{\dagger};$

(iv) $XAA^{\dagger} = X, \quad XA = A^{\#}A;$

(v) $XAX = X, \quad AX = AA^{\dagger}, \quad XA^2 = A;$

(vi) $XAX = X, \quad AXA = A, \quad AX = AA^{\dagger}, \quad XA^2 = A;$

(vii) $XAA^{\dagger} = X, \quad XA^2 = A;$

(viii) $XAA^{\dagger} = X, \quad XAA^* = A^{\#}AA^*.$

Some well-known and several new characterizations of the $*$CEPMP inverse can be obtained using Theorem 2.3.2.

Corollary 2.3.3. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times n}$ *is the* $*$CEPMP *inverse* $A_{\textcircled{\dagger}}AA^{\dagger}$ *of* A*;*

(ii) $XAX = X, \quad AXA = AA_{\textcircled{\dagger}}A, \quad AX = AA_{\textcircled{\dagger}}AA^{\dagger}, \quad XA = A_{\textcircled{\dagger}}A;$

(iii) $A_{\textcircled{\dagger}}AX = X, \quad AX = AA_{\textcircled{\dagger}}AA^{\dagger};$

(iv) $XAA_{\textcircled{\dagger}}AA^{\dagger} = X, \quad XA = A_{\textcircled{\dagger}}A;$

(v) $XAA_{\textcircled{\dagger}}AX = X, \quad AA_{\textcircled{\dagger}}AX = AA_{\textcircled{\dagger}}AA^{\dagger}, \quad XAA_{\textcircled{\dagger}}A = A_{\textcircled{\dagger}}A;$

(v') $XA(A^k)^{\dagger}A^kX = X, \quad A^kX = A^kA^{\dagger}, \quad XA(A^k)^* = (A^k)^*;$

(vi) $XAA_{\textcircled{\dagger}}AX = X, \quad AA_{\textcircled{\dagger}}AXAA_{\textcircled{\dagger}}A = AA_{\textcircled{\dagger}}A,$
$AA_{\textcircled{\dagger}}AX = AA_{\textcircled{\dagger}}AA^{\dagger}, \quad XAA_{\textcircled{\dagger}}A = A_{\textcircled{\dagger}}A;$

(vi') $XA(A^k)^{\dagger}A^kX = X, \quad A^kXA(A^k)^{\dagger}A^k = A^k,$
$A^kX = A^kA^{\dagger}, \quad XA(A^k)^* = (A^k)^*;$

(vii) $A_{\textcircled{\dagger}}AX = X, \quad AA_{\textcircled{\dagger}}AX = AA_{\textcircled{\dagger}}AA^{\dagger};$

(vii') $(A^k)^{\dagger}A^kX = X, \quad A^kX = A^kA^{\dagger};$

(viii) $XAA_{\textcircled{\dagger}}AA^{\dagger} = X, \quad XAA_{\textcircled{\dagger}}A = A_{\textcircled{\dagger}}A;$

(viii') $XA(A^k)^{\dagger}A^kA^{\dagger} = X, \quad XA(A^k)^* = (A^k)^*;$

(ix) $XAA^\dagger = X \quad XAA^* = A_{\textcircled{\dagger}}AA^*$;

(x) $XAA^\dagger = X, \quad XA = A_{\textcircled{\dagger}}A$.

Since the OMP inverse of A is an outer inverse of A, it is interesting to find its range and null space. Also, we can consider projectors involving the OMP inverse.

Lemma 2.3.1. [108, Lemma 2.1] *If $A \in \mathbb{C}^{m\times n}_{T,S}$, then:*

(i) $AA^{(2),\dagger}_{T,S}$ *is a projector onto* $\mathcal{R}(AA^{(2)}_{T,S})$ *along* $\mathcal{N}(A^{(2)}_{T,S}AA^\dagger)$*;*

(ii) $A^{(2),\dagger}_{T,S}A$ *is a projector onto* T *along* $\mathcal{N}(A^{(2)}_{T,S}A)$*;*

(iii) $A^{(2),\dagger}_{T,S} = A^{(2)}_{T,\mathcal{N}(A^{(2)}_{T,S}AA^\dagger)}$.

Proof. (i) Because $A^{(2),\dagger}_{T,S}AA^{(2),\dagger}_{T,S} = A^{(2),\dagger}_{T,S}$, we deduce that $AA^{(2),\dagger}_{T,S}$ is a projector. From $A^{(2),\dagger}_{T,S} = A^{(2)}_{T,S}AA^\dagger$, we get

$$\mathcal{R}(AA^{(2),\dagger}_{T,S}) \subseteq \mathcal{R}(AA^{(2)}_{T,S}) = \mathcal{R}(AA^{(2)}_{T,S}AA^\dagger AA^{(2)}_{T,S}) \subseteq \mathcal{R}(AA^{(2),\dagger}_{T,S}),$$

i.e., $\mathcal{R}(AA^{(2),\dagger}_{T,S}) = \mathcal{R}(AA^{(2)}_{T,S})$. Note that $\mathcal{N}(AA^{(2),\dagger}_{T,S}) = \mathcal{N}(A^{(2),\dagger}_{T,S}) = \mathcal{N}(A^{(2)}_{T,S}AA^\dagger)$.

(ii) We observe that $A^{(2),\dagger}_{T,S}A = A^{(2)}_{T,S}A$. Thus, $\mathcal{N}(A^{(2),\dagger}_{T,S}A) = \mathcal{N}(A^{(2)}_{T,S}A)$ and

$$\mathcal{R}(A^{(2),\dagger}_{T,S}A) = \mathcal{R}(A^{(2)}_{T,S}A) = \mathcal{R}(A^{(2)}_{T,S}) = T.$$

(iii) It follows by $\mathcal{R}(A^{(2),\dagger}_{T,S}) = \mathcal{R}(A^{(2),\dagger}_{T,S}A) = T$ and $\mathcal{N}(A^{(2),\dagger}_{T,S}) = \mathcal{N}(A^{(2)}_{T,S}AA^\dagger)$. □

Using Lemma 2.3.1, we obtain the nullity and range of the DMP inverse as well as some properties of projectors involving the DMP inverse. The parts (i)–(ii) of Corollary 2.3.4 are equivalent to results presented in [86, Theorem 2.12].

Corollary 2.3.4. *If $A \in \mathbb{C}^{n\times n}$ and* $\operatorname{ind}(A) = k$*, then*

(i) $AA^{\mathrm{D},\dagger}$ *is a projector onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}(A^kA^\dagger)$*;*

(ii) $A^{\mathrm{D},\dagger}A$ *is a projector onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}(A^k)$*;*

(iii) $A^{\mathrm{D},\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^\dagger)} = A^{(2)}_{\mathcal{R}(A^kA^\dagger),\mathcal{N}(A^kA^\dagger)}$.

Under the assumption $\operatorname{ind}(A) = 1$ in Corollary 2.3.4, it is possible to conclude the following properties of the core inverse of A, presented in Corollary 2.3.5.

Corollary 2.3.5. *If $A \in \mathbb{C}^{n\times n}$ and* $\operatorname{ind}(A) = 1$*, then*

(i) $AA^{\textcircled{\#}}$ *is the orthogonal projector onto* $\mathcal{R}(A)$*;*

(ii) $A^{\textcircled{\#}}A$ *is a projector onto* $\mathcal{R}(A)$ *along* $\mathcal{N}(A)$*;*

(iii) $A^{\textcircled{\#}} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)} = A^{(2)}_{\mathcal{R}(AA^*),\mathcal{N}(AA^*)}$.

Lemma 2.3.1 gives some properties related to the ∗CEPMP inverse, presented in Corollary 2.3.6.

Corollary 2.3.6. *If $A \in \mathbb{C}^{n\times n}$ and* $\operatorname{ind}(A) = k$*, then:*

(i) $AA_{\textcircled{\dagger},\dagger}$ *is a projector onto* $\mathcal{R}\left(A(A^k)^*\right)$ *along* $\mathcal{N}\left(A^kA^\dagger\right)$*;*

(ii) $A_{\textcircled{\dagger},\dagger}A$ *is a projector onto* $\mathcal{R}((A^k)^*)$ *along* $\mathcal{N}(A^k)$*;*

(iii) $A_{\textcircled{\dagger},\dagger} = A^{(2)}_{\mathcal{R}\left((A^k)^*\right),\mathcal{N}\left(A^kA^\dagger\right)} = A^{(2)}_{\mathcal{R}((A^k)^*A^kA^\dagger),\mathcal{N}((A^k)^*A^kA^\dagger)}$.

From a geometrical point of view, the characterization of the OMP inverse was verified in [108, Theorem 2.8]. Notice that both algebraic and geometrical approaches are equivalent.

Theorem 2.3.4. *If $A \in \mathbb{C}^{m\times n}_{T,S}$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right),\mathcal{N}\left(A^{(2)}_{T,S}AA^\dagger\right)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq T \tag{2.6}$$

is consistent and it has the unique solution $X = A^{(2),\dagger}_{T,S}$.

Proof. By Lemma 2.3.1, $AA_{T,S}^{(2),\dagger} = P_{\mathcal{R}(AA_{T,S}^{(2)}),\mathcal{N}(A_{T,S}^{(2)}AA^\dagger)}$ and $\mathcal{R}(A_{T,S}^{(2),\dagger}) \subseteq T$. So, $A_{T,S}^{(2),\dagger}$ satisfies conditions (2.6).

Suppose that two matrices X and X_1 satisfy (2.6). Then

$$A(X - X_1) = P_{\mathcal{R}(AA_{T,S}^{(2)}),\mathcal{N}(A_{T,S}^{(2)}AA^\dagger)} - P_{\mathcal{R}(AA_{T,S}^{(2)}),\mathcal{N}(A_{T,S}^{(2)}AA^\dagger)} = 0$$

gives $\mathcal{R}(X - X_1) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(A_{T,S}^{(2)}A)$. Because $\mathcal{R}(X) \subseteq T = \mathcal{R}(A_{T,S}^{(2)}) = \mathcal{R}(A_{T,S}^{(2)}A)$ and $\mathcal{R}(X_1) \subseteq \mathcal{R}(A_{T,S}^{(2)}A)$, we observe that $\mathcal{R}(X - X_1) \subseteq \mathcal{R}(A_{T,S}^{(2)}A) \cap \mathcal{N}(A_{T,S}^{(2)}A) = \{0\}$. Hence, $X = X_1$ and the system (2.6) has the unique solution $A_{T,S}^{(2),\dagger}$. □

Applying Theorem 2.3.4, we characterize the DMP inverse from a geometrical point of view. We can observe that Corollary 2.3.7 recovers [86, Theorem 2.13].

Corollary 2.3.7. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}(A^kA^\dagger)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k)$$

is consistent and it has the unique solution $X = A^{\mathrm{D},\dagger}$.

In the case that $\operatorname{ind}(A) = 1$ in Corollary 2.3.7, we get the characterization of the core inverse which coincides with the definition of the core inverse given in [2].

Corollary 2.3.8. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = 1$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}(A)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A)$$

is consistent and it has the unique solution $X = A^{\oplus}$.

Theorem 2.3.4 yields a characterization of the ∗CEPMP inverse from a geometrical point of view.

Corollary 2.3.9. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}\left(A(A^k)^*\right),\mathcal{N}\left(A^kA^\dagger\right)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}((A^k)^*)$$

is consistent and it has the unique solution $X = A_{\textcircled{\dagger},\dagger}$.

Remark that Corollary 2.3.9 is a special case of [19, Theorem 3.3].

2.3.2 Representations of the OMP inverse

In this subsection, we investigate the general, integral and limit representations of the OMP inverse.

The maximal classes of matrices providing the most general form which represents the OMP were developed in [108, Theorem 2.11].

Theorem 2.3.5. [108, Theorem 2.11] *Let $A \in \mathbb{C}_{T,S}^{m\times n}$ and let $U, V \in \mathbb{C}^{n\times m}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A_{T,S}^{(2),\dagger} = UAV$;

(ii) $UA = A_{T,S}^{(2)}A$ *and* $A_{T,S}^{(2)}AA^\dagger = A_{T,S}^{(2)}AV$;

(iii) $\mathcal{R}(UA) = T$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(A_{T,S}^{(2)}AV)$ *and* $AUA = AA_{T,S}^{(2)}A$;

(iv) $U = A_{T,S}^{(2)} + Y(I_m - AA^\dagger)$ *and* $V = A^\dagger + (I_n - A_{T,S}^{(2)}A)Z$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times m}$.

Proof. (i) ⇒ (ii): The equalities $AVA = A$, $A_{T,S}^{(2),\dagger} = UAV$ and $A_{T,S}^{(2),\dagger} = A_{T,S}^{(2)}AA^\dagger$ imply

$$UA = (UAV)A = A_{T,S}^{(2),\dagger}A = A_{T,S}^{(2)}AA^\dagger A = A_{T,S}^{(2)}A$$

and

$$A_{T,S}^{(2)}AA^\dagger = A_{T,S}^{(2),\dagger} = (UA)V = A_{T,S}^{(2)}AV.$$

(ii) ⇒ (iii): The assumption $UA = A_{T,S}^{(2)}A$ yields $AUA = AA_{T,S}^{(2)}A$ and $\mathcal{R}(UA) = \mathcal{R}(A_{T,S}^{(2)}A) = \mathcal{R}(A_{T,S}^{(2)}) = T$. Applying the condition $A_{T,S}^{(2)}AA^\dagger = A_{T,S}^{(2)}AV$, we get $\mathcal{N}(A^*) = \mathcal{N}(A^\dagger) \subseteq \mathcal{N}(A_{T,S}^{(2)}AA^\dagger) = \mathcal{N}(A_{T,S}^{(2)}AV)$.

(iii) $\Rightarrow$ (i): Since $\mathcal{R}(UA) = T = \mathcal{R}(A^{(2)}_{T,S}) = \mathcal{R}(A^{(2)}_{T,S}A)$ and $AUA = AA^{(2)}_{T,S}A$, then

$$UA = A^{(2)}_{T,S}AUA = A^{(2)}_{T,S}AA^{(2)}_{T,S}A = A^{(2)}_{T,S}A.$$

Using $\mathcal{R}(I - AA^\dagger) = \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) \subseteq \mathcal{N}(A^{(2)}_{T,S}AV)$, we obtain

$$A^{(2)}_{T,S}AV = A^{(2)}_{T,S}AVAA^\dagger = A^{(2)}_{T,S}AA^\dagger.$$

Therefore,

$$UAV = A^{(2)}_{T,S}AV = A^{(2)}_{T,S}AA^\dagger = A^{(2),\dagger}_{T,S}.$$

(ii) $\Rightarrow$ (iv): Recall that all solutions of equation $UA = A^{(2)}_{T,S}A$ are obtained as a sum of a particular solution of $UA = A^{(2)}_{T,S}A$ and the general solution of the homogeneous equation $UA = 0$. According to [8, p. 52], the general solution of $UA = A^{(2)}_{T,S}A$ is given by

$$U = A^{(2)}_{T,S} + Y(I_m - AA^\dagger),$$

for arbitrary $Y \in \mathbb{C}^{n\times m}$. In a same way, the general solution of $A^{(2)}_{T,S}AA^\dagger = A^{(2)}_{T,S}AV$ is given by

$$V = A^\dagger + (I_n - A^{(2)}_{T,S}A)Z,$$

for arbitrary $Z \in \mathbb{C}^{n\times m}$.

(iv) $\Rightarrow$ (i): If $U = A^{(2)}_{T,S} + Y(I_m - AA^\dagger)$ and $V = A^\dagger + (I_n - A^{(2)}_{T,S}A)Z$, for arbitrary $Y, Z \in \mathbb{C}^{n\times m}$, then

$$UAV = A^{(2)}_{T,S}AV = A^{(2)}_{T,S}AA^\dagger = A^{(2),\dagger}_{T,S}.$$

□

Theorem 2.3.5 gives maximal classes of matrices determining the DMP inverse. It is interesting to compare assumptions of Corollary 2.3.10 with conditions of [39, Theorem 2.2].

Corollary 2.3.10. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and let $U, V \in \mathbb{C}^{n\times n}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{\mathrm{D},\dagger} = UAV$;

(ii) $UA = A^{\mathrm{D}}A$ *and* $A^kA^\dagger = A^kV$;

(iii) $\mathcal{R}(UA) = \mathcal{R}(A^k)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(A^kV)$ *and* $AUA = AA^{\mathrm{D}}A$;

(iv) $U = A^{\mathrm{D}} + Y(I_n - AA^\dagger)$ *and* $V = A^\dagger + (I_n - A^{\mathrm{D}}A)Z$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times n}$.

Corollary 2.3.10 implies the corresponding result for the core inverse (see also [71, Theorem 3]).

Corollary 2.3.11. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = 1$ and let $U, V \in \mathbb{C}^{n\times n}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{\oplus} = UAV$;

(ii) $UA = A^{\#}A$ *and* $AA^\dagger = AV$;

(iii) $\mathcal{R}(UA) = \mathcal{R}(A)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(AV)$ *and* $AUA = A$;

(iv) $U = A^{\#} + Y(I_n - AA^\dagger)$ *and* $V = A^\dagger + (I_n - A^{\#}A)Z$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times n}$.

In view of Theorem 2.3.5, it is possible to derive some new characterizations of the general representation of the $*$CEPMP inverse.

Corollary 2.3.12. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = 1$ and let $U, V \in \mathbb{C}^{n\times n}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A_{\oplus,\dagger} = UAV$;

(ii) $UA = (A^k)^\dagger A^k$ *and* $A^kA^\dagger = A^kV$;

(iii) $\mathcal{R}(UA) = T$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(A^kV)$ *and* $AUA = A(A^k)^\dagger A^k$;

(iv) $U = A_{\oplus} + Y(I_m - AA^\dagger)$ *and* $V = A^\dagger + (I_n - (A^k)^\dagger A^k)Z$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times m}$.

Recall that, in the case that the eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in the open right halfplane, the inverse of A is represented as

$$A^{-1} = \int_0^\infty \exp(-tA)dt.$$

Some integral representations of different generalized inverses such as the Moore-Penrose and the Drazin inverse were presented in papers [16, 17, 46]. Some of these integral representations impose certain restrictions on eigenvalues of A.

We now state some integral representations for the OMP inverse.

Theorem 2.3.6. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(i) *If $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$, then*

$$\begin{aligned} A^{(2),\dagger}_{T,S} &= \int_0^\infty \exp\big[-G(GAG)^*GAt\big]G(GAG)^*G\mathrm{d}t \int_0^\infty AA^*\exp(-AA^*u)\mathrm{d}u \\ &= \int_0^\infty \exp\big[-G(GAG)^*GAt\big]G(GAG)^*GP_{R(A)}\,\mathrm{d}t. \end{aligned}$$

(ii) *If $G_1 \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G_1) = T$ and $\mathcal{N}(G_1) = \mathcal{N}(A^{(2)}_{T,S}AA^\dagger)$, then*

$$A^{(2),\dagger}_{T,S} = \int_0^\infty \exp\big[-G_1(G_1AG_1)^*G_1At\big]G_1(G_1AG_1)^*G_1\,\mathrm{d}t.$$

Proof. (i) By [165, Theorem 2.2] and [46], we have

$$A^{(2)}_{T,S} = \int_0^\infty \exp\big[-G(GAG)^*GAt\big]G(GAG)^*G\,\mathrm{d}t$$

and

$$A^\dagger = \int_0^\infty A^*\exp(-AA^*u)\mathrm{d}u.$$

The proof of this part can be completed using the definition of the OMP inverse.

(ii) It is well-known, by Lemma 2.3.1(iii), that $A^{(2),\dagger}_{T,S} = A^{(2)}_{T,\mathcal{N}(A^{(2)}_{T,S}AA^\dagger)}$. Applying [165, Theorem 2.2], we obtain that (ii) holds. □

For $G = A^k$ and $G_1 = A^kA^\dagger$ in Theorem 2.3.6, where $\mathrm{ind}(A) = k$, we have integral representations for the DMP inverse.

Corollary 2.3.13. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, then*

$$\begin{aligned} A^{\mathrm{D},\dagger} &= \int_0^\infty \exp\big[-A^k(A^{2k+1})^*A^{k+1}t\big]A^k(A^{2k+1})^*A^k\mathrm{d}t\int_0^\infty AA^*\exp(-AA^*u)\mathrm{d}u \\ &= \int_0^\infty \exp\big[-A^k(A^{2k+1})^*A^{k+1}t\big]A^k(A^{2k+1})^*A^kP_{R(A)}\,\mathrm{d}t \\ &= \int_0^\infty \exp\big[-A^kA^\dagger(A^{2k-1})^*A^kt\big]A^kA^\dagger(A^{2k-1})^*A^kA^\dagger\,\mathrm{d}t. \end{aligned}$$

If $k = 1$ in Corollary 2.3.13, the integral representations for the core inverse can be obtained.

Corollary 2.3.14. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$, then*

$$\begin{aligned} A^{\textcircled{\#}} &= \int_0^\infty \exp\big[-A(A^3)^*A^2t\big]A(A^3)^*A\,\mathrm{d}t\int_0^\infty AA^*\exp(-AA^*u)\mathrm{d}u \\ &= \int_0^\infty \exp\big[-A(A^3)^*A^2t\big]A(A^3)^*AP_{R(A)}\,\mathrm{d}t \\ &= \int_0^\infty \exp\big(-AA^\dagger A^*At\big)AA^\dagger A^*\,\mathrm{d}t. \end{aligned}$$

In the case that $G = (A^k)^\dagger A^k$ and $G_1 = (A^k)^\dagger A^kA^\dagger$ in Theorem 2.3.6, we get new integral representations for the $*$CEPMP inverse.

Corollary 2.3.15. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. Then*

$$\begin{aligned}
A_{\textcircled{\dagger},\dagger} &= \int_0^\infty \exp\left[-A^*(A^k)^\dagger A^{k+1}t\right] A^*(A^k)^\dagger A^k \mathrm{d}t \int_0^\infty AA^* \exp(-AA^*u)\mathrm{d}u \\
&= \int_0^\infty \exp\left[-A^*(A^k)^\dagger A^{k+1}t\right] A^*(A^k)^\dagger A^k P_{R(A)}\mathrm{d}t \\
&= \int_0^\infty \exp\left[-(A^k)^\dagger A^{k-1}((A^k)^\dagger A^\dagger)^* t\right] (A^k)^\dagger A^{k-1}((A^k)^\dagger A^\dagger)^* A^\dagger \mathrm{d}t.
\end{aligned}$$

Notice that investigation of the limit representation of different kinds of generalized inverses is a hot topics many years. One limit representation for Drazin inverse was proved by Meyer [90] in 1974. Some limit representations for outer inverses are given in [76, 131, 161].

Proposition 2.3.1. [161] *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose that $G \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. If $A^{(2)}_{T,S}$ exists, then it possesses the limit representations*

$$A^{(2)}_{T,S} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G (AG + \lambda I)^{-1}. \tag{2.7}$$

Proposition 2.3.2. [76, Theorem 7] *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose that $P \in \mathbb{C}^{n\times s}_s$ satisfies $\mathcal{R}(P) = T$ and $Q \in \mathbb{C}^{s\times m}_s$ satisfies $\mathcal{N}(Q) = S$. If $A^{(2)}_{T,S}$ exists, then*

$$A^{(2)}_{T,S} = \lim_{\lambda\to 0} P (QAP + \lambda I)^{-1} Q. \tag{2.8}$$

The limit representations for the OMP inverse were developed in [108].

Theorem 2.3.7. [108] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$, $B, B_1 \in \mathbb{C}^{n\times s}_s$ and $C, C_1 \in \mathbb{C}^{s\times m}_s$.*

(i) *If $\mathcal{R}(B) = T$ and $\mathcal{N}(C) = S$, then*

$$\begin{aligned}
A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} &= \lim_{t\to 0} B(tI + CAB)^{-1}C \lim_{\lambda\to 0} AA^*(\lambda I + AA^*)^{-1} \\
&= \lim_{t\to 0} B(tI + CAB)^{-1}CP_{\mathcal{R}(A)}.
\end{aligned}$$

In addition, if $\operatorname{rank}(CABC) = \operatorname{rank}(CA)$, then

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}A^\dagger.$$

(ii) *If $\mathcal{R}(B_1) = T$ and $\mathcal{N}(C_1) = N(A^{(2)}_{T,S}AA^\dagger)$, then*

$$\begin{aligned}
A^{(2),\dagger}_{T,S} &= \lim_{t\to 0} B_1(tI + C_1AB_1)^{-1}C_1 \\
&= \lim_{t\to 0}(tI + B_1C_1A)^{-1}B_1C_1 = \lim_{t\to 0} B_1C_1(tI + AB_1C_1)^{-1}.
\end{aligned}$$

Proof. (i) Using Proposition 2.3.2 and [131], we observe that

$$A^{(2)}_{T,S} = \lim_{t\to 0} B(tI + CAB)^{-1}C$$

and

$$A^\dagger = \lim_{\lambda\to 0} A^*(\lambda I + AA^*)^{-1} = \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*.$$

We can easily complete this part of the proof.

Let $\operatorname{rank}(CABC) = \operatorname{rank}(CA)$. Because $\operatorname{rank}(CAB) = \operatorname{rank}(C) = \operatorname{rank}(B) = s$, we have

$$\operatorname{rank}(CA) \le \operatorname{rank}(C) = \operatorname{rank}(CAB) \le \operatorname{rank}(CA).$$

Hence, $\operatorname{rank}(CA) = \operatorname{rank}(C)$ and, by $\mathcal{R}(CA) \subseteq \mathcal{R}(C)$, $\mathcal{R}(CA) = \mathcal{R}(C)$. Since $\operatorname{rank}(CABC) = \operatorname{rank}(CA) = \operatorname{rank}(C) = \operatorname{rank}(B)$, the outer inverse $B^{(2)}_{\mathcal{R}(CA),\mathcal{N}(CA)} = B^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}$ exists. Then the limit representation of the OMP inverse can be transformed using Theorem 2.3.7(i) into

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = \lim_{t\to 0} B(tI + CAB)^{-1}CA\,A^\dagger.$$

Applying Proposition 2.3.1, one can conclude

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(2)}_{\mathcal{R}(CA),\mathcal{N}(CA)}A^\dagger = BB^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}A^\dagger.$$

(ii) It follows by Lemma 2.3.1 and Proposition 2.3.2. $\square$

Theorem 2.3.7 gives limit representations for the DMP, core and $*$CEPMP inverses.

Corollary 2.3.16. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then*

$$\begin{aligned}
A^{\mathrm{D},\dagger} &= \lim_{t\to 0} A^k(tI + A^{2k})^{-1}A^kA^\dagger \\
&= \lim_{t\to 0}(tI + A^{2k})^{-1}A^{2k}A^\dagger = \lim_{t\to 0} A^{2k}A^\dagger(tI + A^{2k+1}A^\dagger)^{-1} \\
&= \lim_{t\to 0} A^k(tI + A^{2k+1})^{-1}A^k \lim_{\lambda\to 0} AA^*(\lambda I + AA^*)^{-1} \\
&= A^k(A^k)^{\#}A^\dagger.
\end{aligned}$$

Corollary 2.3.17. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = 1$, then*

$$\begin{aligned}
A^{\textcircled{\#}} &= \lim_{t\to 0} A(tI + A^2)^{-1}AA^\dagger \\
&= \lim_{t\to 0}(tI + A^2)^{-1}A^2A^\dagger = \lim_{t\to 0} A^2A^\dagger(tI + A^3A^\dagger)^{-1} \\
&= \lim_{t\to 0} A(tI + A^3)^{-1}A \lim_{\lambda\to 0} AA^*(\lambda I + AA^*)^{-1}.
\end{aligned}$$

Corollary 2.3.18. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then*

$$\begin{aligned}
A_{\textcircled{\dagger},\dagger} &= \lim_{t\to 0}(A^k)^*(tI + A^{k+1}(A^k)^*)^{-1}A^k \lim_{\lambda\to 0} AA^*(\lambda I + AA^*)^{-1} \\
&= \lim_{t\to 0}(A^k)^*(tI + A^{k+1}(A^k)^*)^{-1}A^kP_{\mathcal{R}(A)} \\
&= \lim_{t\to 0}(A^k)^*(tI + A^k(A^k)^*)^{-1}A^kA^\dagger \\
&= \lim_{t\to 0}(tI + (A^k)^*A^k)^{-1}(A^k)^*A^kA^\dagger \\
&= \lim_{t\to 0}(A^k)^*A^kA^\dagger(tI + A(A^k)^*A^kA^\dagger)^{-1} \\
&= (A^k)^\dagger A^kA^\dagger.
\end{aligned}$$

2.4 MPO inverses

We now investigate characterizations of the second inverse introduced using the Moore-Penrose inverse and the outer inverse, known as the MPO inverse. The MPO inverse can become the dual core inverse, the MPD inverse and the MPCEP inverse under particular conditions.

2.4.1 Characterizations of the MPO inverse

In this subsection, we present characterizations of the MPO inverse. Applying these results, we characterize the dual core inverse, the MPD inverse and the MPCEP inverse.

The MPO inverse was defined in [108] as a solution of the adequate system of equations.

Theorem 2.4.1. [108] *If $A \in \mathbb{C}^{m\times n}_{T,S}$, the system of equations*

$$XAX = X, \qquad AX = AA^{(2)}_{T,S} \qquad \text{and} \qquad XA = A^\dagger AA^{(2)}_{T,S}A \tag{2.9}$$

is consistent and its unique solution is $X = A^\dagger AA^{(2)}_{T,S}$.

Definition 2.4.1. [108] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The* MPO *(or Moore-Penrose outer) inverse of A is defined as*

$$A^{\dagger,(2)}_{T,S} = A^\dagger AA^{(2)}_{T,S}.$$

The main particular cases of the MPO inverse are listed as follows.

(i) For $m = n$, $\operatorname{ind}(A) = k$ and $A^{(2)}_{T,S} = A^{\mathrm{D}}$, notice that the system (2.9) becomes

$$XAX = X, \qquad AX = AA^{\mathrm{D}} \qquad \text{and} \qquad XA = A^\dagger AA^{\mathrm{D}}A. \tag{2.10}$$

According to Theorem 2.4.1, the MPD inverse $X := A^\dagger AA^{\mathrm{D}} = A^{\dagger,\mathrm{D}}$ is the unique solution to (2.10) and it is a special case of the MPO inverse.

(ii) In the case that $m = n$, $\operatorname{ind}(A) = 1$ and $A^{(2)}_{T,S} = A^{\#}$, by Theorem 2.4.1, the unique solution to the system

$$XAX = X, \qquad AX = AA^{\#} \qquad \text{and} \qquad XA = A^\dagger A$$

is $X = A^\dagger AA^{\#} = A_{\textcircled{\#}}$. So, the MPO inverse coincides with the dual core inverse in such particular case.

(iii) Let $m = n$, $\mathrm{ind}(A) = k$ and $A^{(2)}_{T,S} = A^{\textcircled{\dagger}}$. Then the MPO inverse reduces to the MPCEP inverse [19].

We now state algebraic characterizations of the MPO inverse.

Theorem 2.4.2. *If $A \in \mathbb{C}^{m\times n}_{T,S}$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times m}$ *is the* MPO *inverse* $A^\dagger AA^{(2)}_{T,S}$ *of* A*;*

(ii) $XAX = X, \quad AXA = AA^{(2)}_{T,S}A, \quad AX = AA^{(2)}_{T,S}, \quad XA = A^\dagger AA^{(2)}_{T,S}A;$

(iii) $A^\dagger AA^{(2)}_{T,S}AX = X, \quad AX = AA^{(2)}_{T,S};$

(iv) $XAA^{(2)}_{T,S} = X, \quad XA = A^\dagger AA^{(2)}_{T,S}A;$

(v) $XAA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}, \quad XAA^{(2)}_{T,S}A = A^\dagger AA^{(2)}_{T,S}A;$

(vi) $XAA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}A = AA^{(2)}_{T,S}A,$
$AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}, \quad XAA^{(2)}_{T,S}A = A^\dagger AA^{(2)}_{T,S}A;$

(vii) $A^\dagger AA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S};$

(viii) $XAA^{(2)}_{T,S} = X, \quad XAA^{(2)}_{T,S}A = A^\dagger AA^{(2)}_{T,S}A;$

(ix) $A^\dagger AX = X, \quad A^*AX = A^*AA^{(2)}_{T,S};$

(x) $A^\dagger AX = X, \quad AX = AA^{(2)}_{T,S}.$

If we suppose that T and S are the range and the null spaces of adequate matrices B and C, respectively, we develop the next characterizations of the MPO inverse $A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ as in [108, Theorem 2.5].

Theorem 2.4.3. [108, Theorem 2.5] *If $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times m}$ *is the* MPO *inverse* $A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ *of* A*;*

(ii) $XAB = A^\dagger AB, \quad XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X;$

(iii) $AXAB = AB, \quad A^\dagger AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X;$

(iv) $A^*AXAB = A^*AB, \quad A^\dagger AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X;$

(v) $CAX = C, \quad A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = X;$

(vi) $CAXA = CA, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X;$

(vii) $CAXAA^* = CAA^*, \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X.$

Applying Theorem 2.4.2, necessary and sufficient conditions for a matrix to be the MPD inverse, the dual core inverse and the MPCEP inverse can be verified.

Corollary 2.4.1. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times n}$ *is the* MPD *inverse* $A^\dagger AA^{\mathrm{D}}$ *of* A*;*

(ii) $XAX = X, \quad AXA = AA^{\mathrm{D}}A, \quad AX = AA^{\mathrm{D}}, \quad XA = A^\dagger AA^{\mathrm{D}}A;$

(iii) $A^\dagger AA^{\mathrm{D}}AX = X, \quad AX = AA^{\mathrm{D}};$

(iv) $XAA^{\mathrm{D}} = X, \quad XA = A^\dagger AA^{\mathrm{D}}A;$

(v) $XAA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AX = AA^{\mathrm{D}}, \quad XAA^{\mathrm{D}}A = A^\dagger AA^{\mathrm{D}}A;$

(v') $XAA^{\mathrm{D}}AX = X, \quad A^{k+1}X = A^k, \quad XA^k = A^\dagger A^k;$

(vi) $XAA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AXAA^{\mathrm{D}}A = AA^{\mathrm{D}}A,$
$AA^{\mathrm{D}}AX = AA^{\mathrm{D}}, \quad XAA^{\mathrm{D}}A = A^\dagger AA^{\mathrm{D}}A;$

(vi') $XAA^{\mathrm{D}}AX = X, \quad A^kXA^k = A^{2k-1}, \quad A^{k+1}X = A^k, \quad XA^k = A^\dagger A^k;$

(vii) $A^\dagger AA^{\mathrm{D}}AX = X, \quad A^{k+1}X = A^k;$

(vii') $A^\dagger AA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AX = AA^{\mathrm{D}};$

(viii) $XAA^{\mathrm{D}} = X, \quad XAA^{\mathrm{D}}A = A^\dagger AA^{\mathrm{D}}A;$

(viii') $XAA^{\mathrm{D}} = X, \quad XA^k = A^\dagger A^k;$

(ix) $A^\dagger AX = X, \quad A^*AX = A^*AA^{\mathrm{D}};$

(x) $A^\dagger AX = X, \quad AX = AA^{\mathrm{D}}$.

Corollary 2.4.2. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$, the following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times n}$ is the dual core inverse $A^\dagger AA^\#$ of A;*

(ii) $XAX = X, \quad AXA = A, \quad AX = AA^\#, \quad XA = A^\dagger A$;

(iii) $A^\dagger AX = X, \quad AX = AA^\#$;

(iv) $XAA^\# = X, \quad XA = A^\dagger A$;

(v) $XAX = X, \quad A^2X = A, \quad XA = A^\dagger A$;

(vi) $XAX = X, \quad AXA = A, \quad A^2X = A, \quad XA = A^\dagger A$;

(vii) $A^\dagger AX = X, \quad A^2X = A$;

(viii) $A^\dagger AX = X, \quad A^*AX = A^*AA^\#$.

Corollary 2.4.3. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, the following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times m}$ is the* MPCEP *inverse $A^\dagger AA^{\textcircled{\dagger}}$ of A;*

(ii) $XAX = X, \quad AXA = AA^{\textcircled{\dagger}}A, \quad AX = AA^{\textcircled{\dagger}}, \quad XA = A^\dagger AA^{\textcircled{\dagger}}A$;

(iii) $A^\dagger AA^{\textcircled{\dagger}}AX = X, \quad AX = AA^{\textcircled{\dagger}}$;

(iv) $XAA^{\textcircled{\dagger}} = X, \quad XA = A^\dagger AA^{\textcircled{\dagger}}A$;

(v) $XAA^{\textcircled{\dagger}}AX = X, \quad AA^{\textcircled{\dagger}}AX = AA^{\textcircled{\dagger}}, \quad XAA^{\textcircled{\dagger}} = A^\dagger AA^{\textcircled{\dagger}}$;

(v') $XA^k(A^k)^\dagger AX = X, \quad (A^k)^*AX = (A^k)^*, \quad XA^k = A^\dagger A^k$;

(vi) $XAA^{\textcircled{\dagger}}AX = X, \quad AA^{\textcircled{\dagger}}AXAA^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A,$
$AA^{\textcircled{\dagger}}AX = AA^{\textcircled{\dagger}}, \quad XAA^{\textcircled{\dagger}}A = A^\dagger AA^{\textcircled{\dagger}}A$;

(vi') $XA^k(A^k)^\dagger AX = X, \quad A^k(A^k)^\dagger AXA^k = A^k,$
$(A^k)^*AX = (A^k)^*, \quad XA^k = A^\dagger A^k$;

(vii) $A^\dagger AA^{\textcircled{\dagger}}AX = X, \quad AA^{\textcircled{\dagger}}AX = AA^{\textcircled{\dagger}}$;

(vii') $A^\dagger A^k(A^k)^\dagger AX = X, \quad (A^k)^*AX = (A^k)^*$;

(viii) $XA^k(A^k)^\dagger = X, \quad XA^k = A^\dagger A^k$;

(viii') $XAA^{\textcircled{\dagger}} = X, \quad XAA^{\textcircled{\dagger}}A = A^\dagger AA^{\textcircled{\dagger}}A$;

(ix) $A^\dagger AX = X, \quad A^*AX = A^*AA^{\textcircled{\dagger}}$;

(x) $A^\dagger AX = X, \quad AX = AA^{\textcircled{\dagger}}$.

Projectors determined by the MPO inverse are studied in the following result as well as the range and null space of the MPO inverse.

Lemma 2.4.1. *If $A \in \mathbb{C}^{m\times n}_{T,S}$, then*

(i) *$AA^{\dagger,(2)}_{T,S}$ is a projector onto $\mathcal{R}(AA^{(2)}_{T,S})$ along S;*

(ii) *$A^{\dagger,(2)}_{T,S}A$ is a projector onto $\mathcal{R}(A^\dagger AA^{(2)}_{T,S})$ along $\mathcal{N}(A^{(2)}_{T,S}A)$;*

(iii) $A^{\dagger,(2)}_{T,S} = A^{(2)}_{\mathcal{R}(A^\dagger AA^{(2)}_{T,S}),S}$.

Lemma 2.4.1 gives corresponding properties of the MPD, dual core and MPCEP inverses.

Corollary 2.4.4. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, then:*

(i) *$AA^{\dagger,\mathrm{D}}$ is a projector onto $\mathcal{R}(A^k)$ along $\mathcal{N}(A^k)$;*

(ii) *$A^{\dagger,\mathrm{D}}A$ is a projector onto $\mathcal{R}(A^\dagger A^k)$ along $\mathcal{N}(A^k)$;*

(iii) $A^{\dagger,\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(A^k)} = A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(A^\dagger A^k)}$.

Corollary 2.4.5. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$, then:*

(i) *$AA_{\textcircled{\#}}$ is a projector onto $\mathcal{R}(A)$ along $\mathcal{N}(A)$;*

(ii) *$A_{\textcircled{\#}}A$ is a projector onto $\mathcal{R}(A^*)$ along $\mathcal{N}(A)$;*

(iii) $A_{\textcircled{\#}} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)} = A^{(2)}_{\mathcal{R}(A^*A),\mathcal{N}(A^*A)}$.

Corollary 2.4.6. *If $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, then:*

(i) *$AA^{\dagger,\textcircled{\dagger}}$ is the orthogonal projector onto $\mathcal{R}(A^k)$;*

(ii) *$A^{\dagger,\textcircled{\dagger}}A$ is a projector onto $\mathcal{R}(A^\dagger A^k)$ along $\mathcal{N}((A^k)^*A)$;*

(iii) $A^{\dagger,\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}((A^k)^*)} = A^{(2)}_{\mathcal{R}(A^\dagger A^k(A^k)^*),\mathcal{N}(A^\dagger A^k(A^k)^*)}$.

Parts (i) and (ii) of Corollary 2.4.6 are special cases of parts (i) and (ii) of [19, Lemma 2.1], respectively.

By [108, Theorem 2.8], the MPO inverse is characterized from a geometrical approach.

Theorem 2.4.4. [108, Theorem 2.8] *If $A \in \mathbb{C}^{m\times n}_{T,S}$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}(AA^{(2)}_{T,S}),S} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is consistent and it has the unique solution $X = A^{\dagger,(2)}_{T,S}$.

According to Theorem 2.4.4, we characterize the MPD inverse, the dual core inverse and the MPCEP inverse by a geometrical point of view.

Corollary 2.4.7. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is consistent and it has the unique solution $X = A^{\dagger,\mathrm{D}}$.

Corollary 2.4.8. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = 1$, then the constrained matrix equation*

$$AX = P_{\mathcal{R}(A),\mathcal{N}(A)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is consistent and it has the unique solution $X = A_{\textcircled{\#}}$.

Corollary 2.4.9. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then the system of conditions*

$$AX = P_{\mathcal{R}(A^k)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is consistent and it has the unique solution $X = A^{\dagger,\textcircled{\dagger}}$.

Remark that Corollary 2.4.9 is a particular case of [19, Theorem 2.3].

2.4.2 Representations of the MPO inverse

In this subsections, we give the general, integral and limit representations of the MPO inverse.

The general form for the MPO inverse is given in Theorem 2.4.5.

Theorem 2.4.5. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$ and let $U, V \in \mathbb{C}^{n\times m}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{\dagger,(2)}_{T,S} = VAU$;

(ii) $AU = AA^{(2)}_{T,S}$, $A^\dagger AA^{(2)}_{T,S} = VAA^{(2)}_{T,S}$;

(iii) $\mathcal{N}(AU) = S$, $\mathcal{R}(VAA^{(2)}_{T,S}) \subseteq \mathcal{R}(A^*)$, $AUA = AA^{(2)}_{T,S}A$;

(iv) $U = A^{(2)}_{T,S} + (I_n - A^\dagger A)Y$ *and* $V = A^\dagger + Z(I_m - AA^{(2)}_{T,S})$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times m}$.

Using Theorem 2.4.5, we get representations of the MPD, dual core and MPCEP inverses in the most general form.

Corollary 2.4.10. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and let $U, V \in \mathbb{C}^{n\times n}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{\dagger,\mathrm{D}} = VAU$;

(ii) $AU = AA^{\mathrm{D}}$, $A^\dagger A^k = VA^k$;

(iii) $\mathcal{N}(AU) = \mathcal{N}(A^k)$, $\mathcal{R}(VA^k) \subseteq \mathcal{R}(A^*)$, $AUA = AA^{\mathrm{D}}A$;

(iv) $U = A^{\mathrm{D}} + (I_n - A^\dagger A)Y$ *and* $V = A^\dagger + Z(I_m - AA^{\mathrm{D}})$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times n}$.

Corollary 2.4.11. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = 1$ and let $U, V \in \mathbb{C}^{n\times n}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A_{\textcircled{\#}} = VAU$;

(ii) $AU = AA^{\#}$. $A^\dagger A = VA$;

(iii) $\mathcal{N}(AU) = \mathcal{N}(A)$, $\mathcal{R}(VA) \subseteq \mathcal{R}(A^*)$ *and* $AUA = A$;

(iv) $U = A^{\#} + (I_n - A^\dagger A)Y$ *and* $V = A^\dagger + Z(I_m - AA^{\#})$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times n}$.

Corollary 2.4.12. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and let $U, V \in \mathbb{C}^{n\times n}$ such that $V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{\dagger,\textcircled{\dagger}} = VAU$;

(ii) $AU = A^k(A^k)^\dagger$, $A^\dagger A^k = VA^k$;

(iii) $\mathcal{N}(AU) = \mathcal{N}((A^k)^*)$, $\mathcal{R}(VA^k) \subseteq \mathcal{R}(A^*)$, $AUA = A^k(A^k)^\dagger A$;

(iv) $U = A^{\textcircled{\#}} + (I_n - A^\dagger A)Y$ *and* $V = A^\dagger + Z(I_m - A^k(A^k)^\dagger)$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times n}$.

It is observable that Corollary 2.4.12 is a special case of [19, Theorem 2.7].

The general integral representations for the MPO inverse are presented in Theorem 2.4.6.

Theorem 2.4.6. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(i) *If $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$, then*

$$\begin{aligned} A^{\dagger,(2)}_{T,S} &= \int_0^\infty A^* \exp(-AA^*u)\mathrm{d}u\, A \int_0^\infty \exp\left[-G(GAG)^*GAt\right]G(GAG)^*G\mathrm{d}t \\ &= \int_0^\infty P_{R(A^*)} \exp\left[-\, G(GAG)^*GAt\right]G(GAG)^*G\,\mathrm{d}t. \end{aligned}$$

(ii) *If $G_2 \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G_2) = R(A^\dagger AA^{(2)}_{T,S})$ and $\mathcal{N}(G_2) = S$, then*

$$A^{\dagger,(2)}_{T,S} = \int_0^\infty \exp\left[-\, G_2(G_2AG_2)^*G_2At\right]G_2(G_2AG_2)^*G_2\,\mathrm{d}t.$$

Using Theorem 2.4.6, we can derive integral representations for the MPD, dual core and MPCEP inverses.

Corollary 2.4.13. *If $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$, then*

$$\begin{aligned} A^{\dagger,\mathrm{D}} &= \int_0^\infty A^* \exp(-AA^*u)\mathrm{d}u\, A \int_0^\infty \exp\left[-A^k(A^{2k+1})^*A^{k+1}t\right]A^k(A^{2k+1})^*A^k\mathrm{d}t \\ &= \int_0^\infty P_{R(A^*)} \exp\left[-\, A^k(A^{2k+1})^*A^{k+1}t\right]A^k(A^{2k+1})^*A^k\mathrm{d}t \\ &= \int_0^\infty \exp\left[-\, A^\dagger A^k(A^{2k-1})^*A^\dagger A^{k+1}t\right]A^\dagger A^k(A^{2k-1})^*A^\dagger A^k\mathrm{d}t. \end{aligned}$$

Corollary 2.4.14. *If $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = 1$, then*

$$\begin{aligned} A_{\textcircled{\#}} &= \int_0^\infty A^* \exp(-AA^*u)\mathrm{d}u\, A \int_0^\infty \exp\left[-\, A(A^3)^*A^2t\right]A(A^3)^*A\,\mathrm{d}t \\ &= \int_0^\infty P_{R(A^*)} \exp\left[-\, A(A^3)^*A^2t\right]A(A^3)^*A\,\mathrm{d}t \\ &= \int_0^\infty \exp\left(-\, A^*A^\dagger A^2t\right)A^*A^\dagger A\,\mathrm{d}t. \end{aligned}$$

Corollary 2.4.15. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$.*

$$\begin{aligned} A^{\dagger,\textcircled{\dagger}} &= \int_0^\infty A^* \exp(-AA^*u)\mathrm{d}uA \int_0^\infty \exp\left[-\, A^k(A^k)^\dagger A^*A^{k+1}t\right]A^k(A^k)^\dagger A^*A^k\mathrm{d}t \\ &= \int_0^\infty P_{R(A^*)} \exp\left[-\, A^k(A^k)^\dagger A^*A^{k+1}t\right]A^k(A^k)^\dagger A^*A^k\mathrm{d}t \\ &= \int_0^\infty \exp\left[-\, A^k(A^k)^\dagger A^*A^{k+1}t\right]A^k(A^k)^\dagger A^*A^k\mathrm{d}t. \end{aligned}$$

We also study the limit representations for the MPO inverse.

Theorem 2.4.7. *Let* $A \in \mathbb{C}^{m\times n}_{T,S}$, $B, B_1 \in \mathbb{C}^{n\times s}_s$ *and* $C, C_1 \in \mathbb{C}^{s\times m}_s$.

(i) *If* $\mathcal{R}(B) = T$ *and* $\mathcal{N}(C) = S$, *then*

$$\begin{aligned} A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} &= \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*A \lim_{t\to 0} B(tI + CAB)^{-1}C \\ &= \lim_{t\to 0} P_{\mathcal{R}(A^*)}B(tI + CAB)^{-1}C. \end{aligned}$$

In addition, if $\operatorname{rank}(BCAB) = \operatorname{rank}(AB)$, *then*

$$A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^\dagger C^{(1,2)}_{\mathcal{R}(AB),\mathcal{N}(B)}C.$$

(ii) *If* $\mathcal{R}(B_2) = \mathcal{R}(A^\dagger A A^{(2)}_{T,S})$ *and* $\mathcal{N}(C_2) = S$, *then*

$$\begin{aligned} A^{\dagger,(2)}_{T,S} &= \lim_{t\to 0} B_2(tI + C_2AB_2)^{-1}C_2 \\ &= \lim_{t\to 0}(tI + B_2C_2A)^{-1}B_2C_2 = \lim_{t\to 0} B_2C_2(tI + AB_2C_2)^{-1}. \end{aligned}$$

As consequences of Theorem 2.4.7, we get the limit representations for the MPD, dual core and MPCEP inverses.

Corollary 2.4.16. *If* $A \in \mathbb{C}^{n\times n}$ *and* $\operatorname{ind}(A) = k$, *then*

$$\begin{aligned} A^{\dagger,\mathrm{D}} &= \lim_{t\to 0} A^\dagger A^k(tI + A^{2k})^{-1}A^k \\ &= \lim_{t\to 0}(tI + A^\dagger A^{2k+1})^{-1}A^\dagger A^{2k} = \lim_{t\to 0} A^\dagger A^{2k}(tI + A^{2k})^{-1} \\ &= \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*A \lim_{t\to 0} A^k(tI + A^{2k+1})^{-1}A^k \\ &= A^\dagger(A^k)^{\#}A^k. \end{aligned}$$

Corollary 2.4.17. *If* $A \in \mathbb{C}^{n\times n}$ *and* $\operatorname{ind}(A) = 1$, *then*

$$\begin{aligned} A_{\oplus} &= \lim_{t\to 0} A^\dagger A(tI + A^2)^{-1}A \\ &= \lim_{t\to 0}(tI + A^\dagger A^3)^{-1}A^\dagger A^2 = \lim_{t\to 0} A^\dagger A^2(tI + A^2)^{-1} \\ &= \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*A \lim_{t\to 0} A(tI + A^3)^{-1}A. \end{aligned}$$

Corollary 2.4.18. *If* $A \in \mathbb{C}^{n\times n}$ *and* $\operatorname{ind}(A) = k$, *then*

$$\begin{aligned} A^{\dagger,\textcircled{\dagger}} &= \lim_{t\to 0} A^\dagger A^k(tI + (A^k)^\dagger A^k)^{-1}(A^k)^\dagger \\ &= \lim_{t\to 0}(tI + A^\dagger A^k(A^k)^\dagger A)^{-1}A^\dagger A^k(A^k)^\dagger \\ &= \lim_{t\to 0} A^\dagger A^k(A^k)^\dagger(tI + A^k(A^k)^\dagger)^{-1} \\ &= \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*A \lim_{t\to 0} A^\dagger A^k(tI + (A^k)^*A^k)^{-1}(A^k)^* \\ &= A^\dagger A^k(A^k)^\dagger. \end{aligned}$$

For more representations of the MPCEP inverse see [107].

2.5 MPOMP inverses

The third type of outer inverses defined in terms which include an outer inverse and the Moore-Penrose inverse is the MPOMP inverse. This new outer inverse reduces to the MPCMP inverse in a particular special case.

2.5.1 Characterizations of the MPOMP inverse

Several equivalent conditions for a rectangular complex matrix to be the MPOMP inverse are established in this subsection.

We firstly present the MPOMP inverse by an algebraic approach.

Theorem 2.5.1. *If $A \in \mathbb{C}^{m\times n}_{T,S}$, the system of equations*

$$XAX = X, \quad AX = AA^{(2)}_{T,S}AA^{\dagger}, \quad XA = A^{\dagger}AA^{(2)}_{T,S}A \tag{2.11}$$

is consistent and its unique solution is $X = A^{\dagger}AA^{(2)}_{T,S}AA^{\dagger}$.

Definition 2.5.1. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The* MPOMP *(or Moore-Penrose outer Moore-Penrose) inverse of A is defined as*

$$A^{\dagger,(2),\dagger}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}AA^{\dagger}.$$

Consider the next special MPOMP inverses:

(i) If $m = n$, $\mathrm{ind}(A) = k$ and $A^{(2)}_{T,S} = A^{\mathrm{D}}$, the system (2.11) reduces to

$$XAX = X, \quad AX = AA^{\mathrm{D}}AA^{\dagger}, \quad XA = A^{\dagger}AA^{\mathrm{D}}A, \tag{2.12}$$

which has the unique solution $X = A^{\dagger}AA^{\mathrm{D}}AA^{\dagger} = A^{c,\dagger}$. Hence, the MPOMP becomes the CMP inverse in this case.

(ii) For $m = n$, $\mathrm{ind}(A) = 1$ and $A^{(2)}_{T,S} = A^{\#}$, the unique solution of

$$XAX = X, \quad AX = AA^{\dagger}, \quad XA = A^{\dagger}A$$

is $X = A^{\dagger}AA^{\dagger} = A^{\dagger}$ and so the MPOMP inverse reduces to the MP inverse in this particular choice.

(iii) When $m = n$, $\mathrm{ind}(A) = k$ and $A^{(2)}_{T,S} = A^{\textcircled{\dagger}}$, the MPOMP inverse becomes the MPCEP inverse [19].

Several characterizations of the MPOMP inverse are listed in the following results, restated from [108].

Theorem 2.5.2. [108] *If $A \in \mathbb{C}^{m\times n}_{T,S}$, the following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times m}$ is the* MPOMP *inverse $A^{\dagger}AA^{(2)}_{T,S}AA^{\dagger}$ of A;*

(ii) $XAX = X, \quad AXA = AA^{(2)}_{T,S}A,$
$AX = AA^{(2)}_{T,S}AA^{\dagger}, \quad XA = A^{\dagger}AA^{(2)}_{T,S}A;$

(iii) $A^{\dagger}AA^{(2)}_{T,S}AX = X, \quad AX = AA^{(2)}_{T,S}AA^{\dagger};$

(iv) $XAA^{(2)}_{T,S}AA^{\dagger} = X, \quad XA = A^{\dagger}AA^{(2)}_{T,S}A;$

(v) $XAA^{(2)}_{T,S}AX{=}X, \quad AA^{(2)}_{T,S}AX{=}AA^{(2)}_{T,S}AA^{\dagger}, \quad XAA^{(2)}_{T,S}A{=}A^{\dagger}AA^{(2)}_{T,S}A;$

(vi) $XAA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}A = AA^{(2)}_{T,S}A,$
$AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}AA^{\dagger}, \quad XAA^{(2)}_{T,S}A = A^{\dagger}AA^{(2)}_{T,S}A;$

(vii) $A^{\dagger}AA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}AA^{\dagger};$

(viii) $XAA^{(2)}_{T,S}AA^{\dagger} = X, \quad XAA^{(2)}_{T,S}A = A^{\dagger}AA^{(2)}_{T,S}A;$

(ix) $A^{\dagger}AXAA^{\dagger} = X, \quad A^{*}AXAA^{*} = A^{*}AA^{(2)}_{T,S}AA^{*};$

(x) $A^{\dagger}AXAA^{\dagger} = X, \quad AXA = AA^{(2)}_{T,S}A.$

Some characterizations of the MPOMP inverse $A^{\dagger}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger}$ are presented in Theorem 2.5.3 .

Theorem 2.5.3. *If $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$, the following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times m}$ is the* MPOMP *inverse $A^{\dagger}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger}$ of A;*

(ii) $XAB = A^{\dagger}AB, \quad XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger} = X;$

(iii) $AXAB = AB, \quad A^{\dagger}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger} = X;$

(iv) $A^{*}AXAB = A^{*}AB, \quad A^{\dagger}AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger} = X;$

(v) $CAX = CAA^{\dagger}, \quad A^{\dagger}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = X;$

(vi) $CAXA = CA, \quad A^{\dagger}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^{\dagger} = X;$

(vii) $CAXAA^* = CAA^*, \quad A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X.$

By Theorem 2.5.2, the following characterizations of the CMP inverse are developed. Some of these characterizations were involved in [177, Theorem 2.3].

Corollary 2.5.1. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, the following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times n}$ *is the* CMP *inverse* $A^\dagger AA^{\mathrm{D}}AA^\dagger$ *of* A*;*

(ii) $XAX = X, \quad AXA = AA^{\mathrm{D}}A,$
$AX = AA^{\mathrm{D}}AA^\dagger, \quad XA = A^\dagger AA^{\mathrm{D}}A;$

(iii) $A^\dagger AA^{\mathrm{D}}AX = X, \quad AX = AA^{\mathrm{D}}AA^\dagger;$

(iv) $XAA^{\mathrm{D}}AA^\dagger = X, \quad XA = A^\dagger AA^{\mathrm{D}}A;$

(v) $XAA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AX = AA^{\mathrm{D}}AA^\dagger, \quad XAA^{\mathrm{D}}A = A^\dagger AA^{\mathrm{D}}A;$

(v') $XAA^{\mathrm{D}}AX = X, \quad A^kX = A^kA^\dagger, \quad XA^k = A^\dagger A^k;$

(vi) $XAA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AXAA^{\mathrm{D}}A = AA^{\mathrm{D}}A,$
$AA^{\mathrm{D}}AX = AA^{\mathrm{D}}AA^\dagger, \quad XAA^{\mathrm{D}}A = A^\dagger AA^{\mathrm{D}}A;$

(vi') $XAA^{\mathrm{D}}AX = X, \quad A^kXA^k = A^{2k-1},$
$A^kX = A^kA^\dagger, \quad XA^k = A^\dagger A^k;$

(vii) $A^\dagger AA^{\mathrm{D}}AX = X, \quad AA^{\mathrm{D}}AX = AA^{\mathrm{D}}AA^\dagger;$

(vii') $A^\dagger AA^{\mathrm{D}}AX = X, \quad A^kX = A^kA^\dagger;$

(viii) $XAA^{\mathrm{D}}AA^\dagger = X, \quad XAA^{\mathrm{D}}A = A^\dagger AA^{\mathrm{D}}A;$

(viii') $XAA^{\mathrm{D}}AA^\dagger = X, \quad XA^k = A^\dagger A^k;$

(ix) $A^\dagger AXAA^\dagger = X, \quad A^*AXAA^* = A^*AA^{\mathrm{D}}AA^*;$

(x) $A^\dagger AXAA^\dagger = X, \quad AXA = AA^{\mathrm{D}}A.$

Lemma 2.5.1 investigates the range and null space of the MPOMP inverse as well as the projectors which contain the MPOMP inverse.

Lemma 2.5.1. *If $A \in \mathbb{C}^{m\times n}_{T,S}$, then*

(i) $AA^{\dagger,(2),\dagger}_{T,S}$ *is a projector onto* $\mathcal{R}(AA^{(2)}_{T,S})$ *along* $\mathcal{N}(A^{(2)}_{T,S}AA^\dagger)$*;*

(ii) $A^{\dagger,(2),\dagger}_{T,S}A$ *is a projector onto* $\mathcal{R}(A^\dagger AA^{(2)}_{T,S})$ *along* $\mathcal{N}(A^{(2)}_{T,S}A)$*;*

(iii) $A^{\dagger,(2),\dagger}_{T,S} = A^{(2)}_{\mathcal{R}\left(A^\dagger AA^{(2)}_{T,S}\right),\mathcal{N}\left(A^{(2)}_{T,S}AA^\dagger\right)}$.

As a consequence of Lemma 2.5.1, we describe the projectors determined by the CMP inverse and also the range and the null space of the CMP inverse.

Corollary 2.5.2. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then*

(i) $AA^{c,\dagger}$ *is a projector onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}(A^kA^\dagger)$*;*

(ii) $A^{c,\dagger}A$ *is a projector onto* $\mathcal{R}(A^\dagger A^k)$ *along* $\mathcal{N}(A^k)$*;*

(iii) $A^{c,\dagger} = A^{(2)}_{\mathcal{R}\left(A^\dagger A^k\right),\mathcal{N}\left(A^kA^\dagger\right)}$.

We give one more characterization of the MPOMP inverse proved in [108, Theorem 2.8].

Theorem 2.5.4. *If $A \in \mathbb{C}^{m\times n}_{T,S}$, then the system of conditions*

$$AX = P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right),\mathcal{N}\left(A^{(2)}_{T,S}AA^\dagger\right)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is consistent and it has the unique solution $X = A^{\dagger,(2),\dagger}_{T,S}$.

Applying Theorem 2.5.4, we present a characterization of the CMP inverse, which recovers [100, Corollary 2.3].

Corollary 2.5.3. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then the system of conditions*

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}(A^kA^\dagger)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is consistent and it has the unique solution $X = A^{c,\dagger}$.

Necessary and sufficient conditions which include the OMP, MPO and MPOMP inverses and ensure that the equality $AA^{(2)}_{T,S}A = A$ holds, are given in the next result. It is interesting to note that $A^{(2)}_{T,S} \in A\{1\} \iff A^{(2),\dagger}_{T,S} \in A\{1\} \iff A^{\dagger,(2)}_{T,S} \in A\{1\} \iff A^{\dagger,(2),\dagger,}_{T,S} \in A\{1\}$.

Theorem 2.5.5. *Let A, T, S satisfy the assumptions of Theorem 2.3.1. The following statements are equivalent:*

(i) $AA^{(2)}_{T,S}A = A$;

(ii) $AA^{(2),\dagger}_{T,S} = AA^\dagger$;

(iii) $A^{\dagger,(2)}_{T,S}A = A^\dagger A$;

(iv) $AA^{\dagger,(2),\dagger}_{T,S} = AA^\dagger$;

(v) $A^{\dagger,(2),\dagger}_{T,S}A = A^\dagger A$;

(vi) $A^{\dagger,(2),\dagger}_{T,S} = A^\dagger$;

(vii) $AA^{(2),\dagger}_{T,S}A = A$;

(viii) $AA^{\dagger,(2)}_{T,S}A = A$;

(ix) $AA^{\dagger,(2),\dagger}_{T,S}A = A$.

Proof. (i) $\Leftrightarrow$ (ii): From $A^{(2),\dagger}_{T,S} = A^{(2)}_{T,S}AA^\dagger$, we deduce that $AA^{(2),\dagger}_{T,S} = AA^\dagger$ is equivalent to $AA^{(2)}_{T,S}AA^\dagger = AA^\dagger$. Multiplying the equality $AA^{(2)}_{T,S}AA^\dagger = AA^\dagger$ by A from the right hand side, we obtain $AA^{(2)}_{T,S}A = A$. On the other hand, multiplying $AA^{(2)}_{T,S}A = A$ by $A^\dagger$ from the right hand side, we get $AA^{(2)}_{T,S}AA^\dagger = AA^\dagger$. Thus, (i) and (ii) are equivalent.

In a similar manner, we show that (i) is equivalent to (iii)-(ix). $\square$

We also can present the following relations between some projectors involving the OMP, MPO and MPOMP inverses.

Theorem 2.5.6. *Let A, T, S satisfy the assumptions of Theorem 2.3.1. Then*

(i) $A^{(2),\dagger}_{T,S} = A^\dagger$ *if and only if* $A^{(2)}_{T,S}A = A^\dagger A$ *if and only if* $A^{(2),\dagger}_{T,S}A = A^\dagger A$;

(ii) $A^{(2),\dagger}_{T,S} = A^*$ *if and only if* $A^{(2)}_{T,S}A = A^*A$ *if and only if* $A^{(2),\dagger}_{T,S}A = A^*A$;

(iii) $A^{\dagger,(2)}_{T,S} = A^\dagger$ *if and only if* $AA^{(2)}_{T,S} = AA^\dagger$ *if and only if* $AA^{\dagger,(2)}_{T,S} = AA^\dagger$;

(iv) $A^{\dagger,(2)}_{T,S} = A^*$ *if and only if* $AA^{(2)}_{T,S} = AA^*$ *if and only if* $AA^{\dagger,(2)}_{T,S} = AA^*$;

(v) $A^{\dagger,(2),\dagger}_{T,S} = A^*$ *if and only if* $AA^{(2)}_{T,S}A = AA^*A$;

(vi) $A^{(2),\dagger}_{T,S} = 0$ *if and only if* $A^{(2)}_{T,S} = 0$ *if and only if* $A^{\dagger,(2)}_{T,S} = 0$ *if and only if* $A^{\dagger,(2),\dagger}_{T,S} = 0$.

Proof. (i) Note that, by $A^{(2),\dagger}_{T,S}A = A^{(2)}_{T,S}A$,

$$A^{(2),\dagger}_{T,S} = A^\dagger \;\Leftrightarrow\; A^{(2)}_{T,S}AA^\dagger = A^\dagger \;\Leftrightarrow\; A^{(2)}_{T,S}A = A^\dagger A \;\Leftrightarrow\; A^{(2),\dagger}_{T,S}A = A^\dagger A.$$

(ii) This part can be proved similarly as part (i), using $A^* = A^*AA^\dagger$.

The rest of the proof follows analogously. $\square$

2.5.2 Representations of the MPOMP inverse

The general form, the integral and limit representations of the MPOMP inverse are presented in this subsection.

We give maximal classes of complex matrices for which the representation of the MPOMP inverse is still valid.

Theorem 2.5.7. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$ and let $Q, U, V \in \mathbb{C}^{n\times m}$ such that $Q, V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{\dagger,(2),\dagger}_{T,S} = QAUAV$;

(ii) $AUA = AA^{(2)}_{T,S}A$, $QAA^{(2)}_{T,S}A = A^\dagger AA^{(2)}_{T,S}A$, $AA^{(2)}_{T,S}AV = AA^{(2)}_{T,S}AA^\dagger$;

(iii) $\mathcal{R}(QAA^{(2)}_{T,S}A) \subseteq \mathcal{R}(A^*)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(AA^{(2)}_{T,S}AV)$, $AUA = AA^{(2)}_{T,S}A$;

(iv) $U = A^\dagger AA^{(2)}_{T,S}AA^\dagger + Y - A^\dagger AYAA^\dagger$, $Q = A^\dagger + Z(I_m - AA^{(2)}_{T,S})$ *and* $V = A^\dagger + (I_n - A^{(2)}_{T,S}A)W$, *for arbitrary* $Y, Z, W \in \mathbb{C}^{n\times m}$.

By Theorem 2.5.7, we obtain the next result about the CMP inverse which generalizes [39, Theorem 4.1].

Corollary 2.5.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and let $Q, U, V \in \mathbb{C}^{n\times n}$ such that $Q, V \in A\{1\}$. Then the following statements are equivalent:*

(i) $A^{c,\dagger} = QAUAV$;

(ii) $AUA = AA^{\mathrm{D}}A$, $QA^k = A^\dagger A^k$, $A^kV = A^kA^\dagger$;

(iii) $\mathcal{R}(QA^k) \subseteq \mathcal{R}(A^*)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(A^kV)$, $AUA = AA^{\mathrm{D}}A$;

(iv) $U = A^\dagger AA^{\mathrm{D}}AA^\dagger + Y - A^\dagger AYAA^\dagger$, $Q = A^\dagger + Z(I_n - AA^{\mathrm{D}})$ *and* $V = A^\dagger + (I_n - A^{\mathrm{D}}A)W$, *for arbitrary* $Y, Z, W \in \mathbb{C}^{n\times n}$.

Integral representations of the MPOMP inverse are given in Theorem 2.5.8.

Theorem 2.5.8. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(i) *If $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$, then*

$$\begin{aligned} A^{\dagger,(2),\dagger}_{T,S} &= \int_0^\infty A^* \exp(-AA^*u)\mathrm{d}u \; A \int_0^\infty \exp\left[-G(GAG)^*GAt\right]G(GAG)^*G\,\mathrm{d}t \\ &\quad \times \int_0^\infty AA^* \exp(-AA^*v)\mathrm{d}v \\ &= \int_0^\infty P_{R(A^*)} \exp\left[-G(GAG)^*GAt\right]G(GAG)^*G\,P_{R(A)}\mathrm{d}t. \end{aligned}$$

(ii) *If $G_3 \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G_3) = \mathcal{R}(A^\dagger AA^{(2)}_{T,S})$ and $\mathcal{N}(G_3) = \mathcal{N}(A^{(2)}_{T,S}AA^\dagger)$, then*

$$A^{\dagger,(2)}_{T,S} = \int_0^\infty \exp\left[-G_3(G_3AG_3)^*G_3At\right]G_3(G_3AG_3)^*G_3\,\mathrm{d}t.$$

By Theorem 2.5.8, we propose integral representations for the CMP inverse.

Corollary 2.5.5. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then*

$$\begin{aligned} A^{c,\dagger} &= \int_0^\infty A^* \exp(-AA^*u)\mathrm{d}u \\ &\quad \times A \int_0^\infty \exp\left[-A^k(A^{2k+1})^*A^{k+1}t\right]A^k(A^{2k+1})^*A^k\,\mathrm{d}t \\ &\quad \times \int_0^\infty AA^* \exp(-AA^*v)\mathrm{d}v \\ &= \int_0^\infty P_{R(A^*)} \exp\left[-A^k(A^{2k+1})^*A^{k+1}t\right]A^k(A^{2k+1})^*A^k\,P_{R(A)}\,\mathrm{d}t \\ &= \int_0^\infty \exp\left[-A^\dagger A^{k-1}(A^\dagger A^{2k-1}A^\dagger)^*A^{k-1}t\right]A^\dagger A^k(A^\dagger A^{2k-1}A^\dagger)^*A^{k-1}\mathrm{d}t. \end{aligned}$$

The limit representations for the MPOMP inverse are stated in the following theorem.

Theorem 2.5.9. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$, $B, B_1 \in \mathbb{C}^{n\times s}_s$ and $C, C_1 \in \mathbb{C}^{s\times m}_s$.*

(i) *If $\mathcal{R}(B) = T$ and $\mathcal{N}(C) = S$, then*

$$\begin{aligned} A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} &= \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*A \lim_{t\to 0} B(tI + CAB)^{-1}C \\ &\quad \times \lim_{\lambda\to 0} AA^*(\lambda I + AA^*)^{-1} \\ &= \lim_{t\to 0} P_{\mathcal{R}(A^*)}B(tI + CAB)^{-1}CP_{\mathcal{R}(A)}. \end{aligned}$$

In addition, if $\operatorname{rank}(CABC) = \operatorname{rank}(CA)$ *and* $\operatorname{rank}(BCAB) = \operatorname{rank}(AB)$, *then*

$$\begin{aligned} A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} &= A^\dagger C^{(1,2)}_{\mathcal{R}(AB),\mathcal{N}(B)}CP_{\mathcal{R}(A)} \\ &= P_{\mathcal{R}(A^*)}B\,B^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}A^\dagger. \end{aligned}$$

(ii) *If* $\mathcal{R}(B_3) = \mathcal{R}(A^\dagger A A^{(2)}_{T,S})$ *and* $\mathcal{N}(C_3) = \mathcal{N}(A^{(2)}_{T,S} A A^\dagger)$*, then*

$$\begin{aligned} A^{\dagger,(2),\dagger}_{T,S} &= \lim_{t\to 0} B_3(tI + C_3 A B_3)^{-1} C_3 \\ &= \lim_{t\to 0}(tI + B_3C_3A)^{-1}B_3C_3 = \lim_{t\to 0} B_3C_3(tI + AB_3C_3)^{-1}. \end{aligned}$$

Corollary 2.5.6. *If* $A \in \mathbb{C}^{n\times n}$ *and* $\mathrm{ind}(A) = k$*, then*

$$\begin{aligned} A^{c,\dagger} &= \lim_{t\to 0} A^\dagger A^k (tI + A^{2k-1})^{-1} A^k A^\dagger \\ &= \lim_{t\to 0}(tI + A^\dagger A^{2k})^{-1} A^\dagger A^{2k} A^\dagger = \lim_{t\to 0} A^\dagger A^{2k} A^\dagger (tI + A^{2k}A^\dagger)^{-1} \\ &= \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*A \lim_{t\to 0} A^k(tI + A^{2k+1})^{-1}A^k \lim_{\lambda\to 0} AA^*(\lambda I + AA^*)^{-1} \\ &= A^\dagger (A^k)^{\#} A^k A^\dagger P_{\mathcal{R}(A)} \\ &= P_{\mathcal{R}(A^*)} A^k (A^k)^{\#} A^\dagger. \end{aligned}$$

Notice that some integral and limit representations of the CMP inverse proved in [96] are recovered in Corollary 2.5.6.

2.6 Examples about OMP and MPO inverses

This section is aimed to numerical illustration of presented results.

Example 2.6.1. *Consider the matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -a & a & a \\ -a & -a & -a \\ 2 & 0 & 1 \end{bmatrix}, a \in \mathbb{R},$$

in conjunction with

$$B = \begin{bmatrix} 1 & 0 \\ a & 0 \\ 0 & a \end{bmatrix}, \ C = \begin{bmatrix} 1 & a & a & 1 \\ 1 & a & a & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

These matrices satisfy $\mathrm{rank}(A) = 3$, $\mathrm{rank}(B) = \mathrm{rank}(C) = \mathrm{rank}(CAB) = 2$*, which is a guarantee for the existence of*

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = B(CAB)^\dagger C.$$

Symbolic calculation gives

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \begin{bmatrix} \frac{1}{-2a^2-a+2} & -\frac{a}{2a^2+a-2} & \frac{-a^2+a-1}{(a-1)(2a^2+a-2)} & -\frac{a}{(a-1)(2a^2+a-2)} \\ -\frac{a}{2a^2+a-2} & -\frac{a^2}{2a^2+a-2} & \frac{a(-a^2+a-1)}{(a-1)(2a^2+a-2)} & -\frac{a^2}{(a-1)(2a^2+a-2)} \\ \frac{a+2}{2a^2+a-2} & \frac{a(a+2)}{2a^2+a-2} & \frac{a^3-a^2-2a+4}{(a-1)(2a^2+a-2)} & \frac{-a^2+a+2}{(a-1)(2a^2+a-2)} \end{bmatrix}.$$

Now, the OMP inverse of A is equal to

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} A A^\dagger =$$
$$\begin{bmatrix} \frac{2a^3-2a^2-a+2}{2a^5-a^4+a^3-6a+4} & \frac{-a^4+a^3+a^2-a+1}{2a^5-a^4+a^3-6a+4} & -\frac{a^4-a^3+a+1}{2a^5-a^4+a^3-6a+4} & -\frac{a}{(a-1)(2a^2+a-2)} \\ \frac{a(2a^3-2a^2-a+2)}{2a^5-a^4+a^3-6a+4} & \frac{-a^5+a^4+a^3-a^2+a}{2a^5-a^4+a^3-6a+4} & -\frac{a(a^4-a^3+a+1)}{2a^5-a^4+a^3-6a+4} & -\frac{a^2}{(a-1)(2a^2+a-2)} \\ -\frac{2(a^4-3a^2+a+2)}{2a^5-a^4+a^3-6a+4} & \frac{a^5+a^4-3a^3+a^2+2a-4}{(a-1)(a^2+2)(2a^2+a-2)} & \frac{a^5-a^4-3a^3+a^2+2a+4}{2a^5-a^4+a^3-6a+4} & \frac{-a^2+a+2}{(a-1)(2a^2+a-2)} \end{bmatrix}.$$

Further, the MPO inverse of A is equal to

$$A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^\dagger A A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} =$$
$$\begin{bmatrix} \frac{1}{-2a^2-a+2} & -\frac{a}{2a^2+a-2} & \frac{-a^2+a-1}{(a-1)(2a^2+a-2)} & -\frac{a}{(a-1)(2a^2+a-2)} \\ -\frac{a}{2a^2+a-2} & -\frac{a^2}{2a^2+a-2} & -\frac{a(a^2-a+1)}{(a-1)(2a^2+a-2)} & -\frac{a^2}{(a-1)(2a^2+a-2)} \\ \frac{a+2}{2a^2+a-2} & \frac{a(a+2)}{2a^2+a-2} & \frac{a^3-a^2-2a+4}{(a-1)(2a^2+a-2)} & \frac{-a^2+a+2}{(a-1)(2a^2+a-2)} \end{bmatrix}.$$

and the MPOMP inverse is

$$A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^\dagger A A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} A A^\dagger =$$

$$\begin{bmatrix} \frac{2a^3-2a^2-a+2}{2a^5-a^4+a^3-6a+4} & \frac{-a^4+a^3+a^2-a+1}{2a^5-a^4+a^3-6a+4} & -\frac{a^4-a^3+a+1}{2a^5-a^4+a^3-6a+4} & -\frac{a}{(a-1)(2a^2+a-2)} \\ \frac{a(2a^3-2a^2-a+2)}{2a^5-a^4+a^3-6a+4} & \frac{-a^5+a^4+a^3-a^2+a}{2a^5-a^4+a^3-6a+4} & -\frac{a(a^4-a^3+a+1)}{2a^5-a^4+a^3-6a+4} & -\frac{a^2}{(a-1)(2a^2+a-2)} \\ -\frac{2(a^4-3a^2+a+2)}{2a^5-a^4+a^3-6a+4} & \frac{a^5+a^4-3a^3+a^2+2a-4}{(a-1)(a^2+2)(2a^2+a-2)} & \frac{a^5-a^4-3a^3+a^2+2a+4}{2a^5-a^4+a^3-6a+4} & \frac{-a^2+a+2}{(a-1)(2a^2+a-2)} \end{bmatrix}.$$

After simplifications, it can be verified that the MPO inverse, OMP as well as MPOMP inverse satisfy the Penrose equation (2).

Example 2.6.2. *Consider the matrix*

$$A = \begin{bmatrix} -0.892519 & 0.625642 & 0.175331 & 0.321944 & -0.232234 \\ 0.365003 & -0.163061 & -1.49664 & 0.739601 & 1.04074 \\ -1.01684 & 0.739323 & -0.190166 & 0.615956 & 0.00524681 \\ -0.139044 & 0.0979357 & -0.327397 & 0.0538087 & -0.0193572 \end{bmatrix},$$

of rank 3 *and matrices*

$$B = \begin{bmatrix} 0.391719 & -0.920874 \\ 0.109785 & -0.124002 \\ -0.607216 & -0.559493 \\ -0.320068 & 0.835002 \\ 0.984287 & -0.742661 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.539054 & 0.766478 & 0.155971 & 0.414893 \\ -0.373504 & 0.163028 & -0.114286 & -0.0928122 \\ -0.255181 & 0.125827 & -0.07821 & -0.0593588 \end{bmatrix}$$

of rank 2. *Since* $\operatorname{rank}(A) = 3$, $\operatorname{rank}(B) = \operatorname{rank}(C) = \operatorname{rank}(CAB) = 2$, *the existence of* $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = B(CAB)^\dagger C$ *is guaranteed. Numerical calculation gives*

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \begin{bmatrix} 0.996462 & 1.24348 \\ 0.59948 & -0.5468 \\ 0.436414 & -0.367027 \end{bmatrix}.$$

Now, the OMP inverse of A is equal to

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} A A^\dagger =$$

$$\begin{bmatrix} -0.301166 & 0.0326154 & -0.360323 & -0.219517 \\ -0.0483605 & 0.0318106 & -0.0507907 & -0.0246206 \\ -0.0673006 & -0.386489 & -0.185273 & -0.206558 \\ 0.268275 & -0.0126901 & 0.325324 & 0.202088 \\ -0.334358 & 0.347601 & -0.317199 & -0.11916 \end{bmatrix}.$$

Further, the MPO inverse of A is equal to

$$A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^\dagger A A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} =$$

$$\begin{bmatrix} -0.484834 & 0.0284051 & -0.14671 & -0.171859 \\ 0.33207 & -0.00229212 & 0.10033 & 0.122522 \\ -0.209065 & -0.425538 & -0.0593427 & -0.196884 \\ 0.10084 & 0.154856 & 0.0290746 & 0.0808309 \\ -0.19085 & 0.19214 & -0.0593712 & -0.0169011 \end{bmatrix}.$$

and the MPOMP inverse is

$$A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^\dagger A A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} A A^\dagger =$$

$$\begin{bmatrix} -0.273048 & 0.0763509 & -0.314237 & -0.18031 \\ 0.184742 & -0.0356455 & 0.21687 & 0.128401 \\ -0.0597676 & -0.391739 & -0.17744 & -0.202841 \\ 0.035502 & 0.140065 & 0.0807583 & 0.0834382 \\ -0.131446 & 0.205588 & -0.106361 & -0.0192716 \end{bmatrix}.$$

Since the Frobenius norm

$$\|A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} A A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}\|_F = 7.71123 * 10^{-16}$$

is near zero, it follows that the OMP inverse is an outer inverse. Similar situation appears with MPO and MPOMP inverses:

$$\|A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}\|_F = 4.52153 * 10^{-16}$$

$$\|A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}\|_F = 7.66812 * 10^{-16}.$$

In order to verify part (ii) of Theorem 2.3.3, we calculate

$$CAA^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - CAA^{\dagger}$$
$$= \begin{bmatrix} 1.1102 * 10^{-16} & 1.1102 * 10^{-16} & 1.110210^{-16} & 4.44089 * 10^{-16} \\ 2.77556 * 10^{-17} & -2.77556 * 10^{-17} & 8.32667 * 10^{-17} & -8.32667 * 10^{-17} \\ 8.32667 * 10^{-17} & -5.5511 * 10^{-17} & 1.38778 * 10^{-16} & -1.11022 * 10^{-16} \end{bmatrix}.$$

Example 2.6.3. *Consider the matrix*

$$A = \begin{bmatrix} 0.101312 & 0.0100717 & -0.573253 & -0.643856 \\ -0.408473 & 0.7352 & 1.3185 & 0.51637 \\ 0.897308 & -0.72165 & -0.0984336 & 0.837847 \end{bmatrix},$$

of rank 3 and matrices

$$B = \begin{bmatrix} 0.207101 + 0.408489i & 0.899214 - 0.327931i \\ -0.0828035 + 0.993832i & 0.0705601 + 0.443512i \\ 0.00881961 - 0.663084i & 0.228649 - 0.730613i \\ 0.478787 - 0.800084i & 0.44478 + 0.407056i \end{bmatrix},$$

$$C = \begin{bmatrix} -0.0442875 + 0.0547413i & -0.162671 + 0.280591i & -0.350576 - 0.156153i \\ -0.160937 + 0.14094i & 0.160623 + 0.254029i & -0.32686 - 0.250027i \\ 0.336677 - 0.0543376i & -0.450241 - 0.584124i & 0.14928 + 0.252637i \\ -0.695626 - 0.556272i & -1.1618 + 1.26054i & 0.399886 - 1.54995i \end{bmatrix}$$

of rank 2. Since $\mathrm{rank}(A) = 3$, $\mathrm{rank}(B) = \mathrm{rank}(C) = \mathrm{rank}(CAB) = 2$, $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = B(CAB)^{\dagger}C$ *exists, and it is equal to*

$$\begin{bmatrix} 0.334828 - 1.10851i & -3.02792 - 0.854075i & 1.77737 - 0.973814i \\ 0.577164 - 0.381528i & -1.3846 - 1.02733i & 0.698604 + 0.182486i \\ -0.450406 - 0.207286i & -0.419421 + 0.72282i & 0.104675 - 0.902493i \\ 0.0973201 + 0.41839i & 1.87974 - 0.542228i & 0.0570891 + 1.09417i \end{bmatrix}.$$

Now, the OMP inverse of A is equal to

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger} =$$
$$\begin{bmatrix} 0.334828 - 1.10851i & -3.02792 - 0.854075i & 1.77737 - 0.973814i \\ 0.577164 - 0.381528i & -1.3846 - 1.02733i & 0.698604 + 0.182486i \\ -0.450406 - 0.207286i & -0.419421 + 0.72282i & 0.104675 - 0.902493i \\ 0.0973201 + 0.41839i & 1.87974 - 0.542228i & 0.0570891 + 1.09417i \end{bmatrix}.$$

Further, the MPO inverse of A is equal to

$$A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{\dagger}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} =$$
$$\begin{bmatrix} 0.348491 - 1.10632i & -3.02046 - 0.881221i & 1.78521 - 0.949198i \\ 0.143976 - 0.451035i & -1.62131 - 0.166675i & 0.450141 - 0.597953i \\ -0.070264 - 0.14629i & -0.211697 - 0.0324428i & 0.322714 - 0.217621i \\ -0.245764 + 0.36334i & 1.69226 + 0.139407i & -0.139693 + 0.476063i \end{bmatrix}.$$

and the MPOMP inverse is

$$A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{\dagger}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{\dagger} =$$
$$\begin{bmatrix} 0.348491 - 1.10632i & -3.02046 - 0.881221i & 1.78521 - 0.949198i \\ 0.143976 - 0.451035i & -1.62131 - 0.166675i & 0.450141 - 0.597953i \\ -0.070264 - 0.14629i & -0.211697 - 0.0324428i & 0.322714 - 0.217621i \\ -0.245764 + 0.36334i & 1.69226 + 0.139407i & -0.139693 + 0.476063i \end{bmatrix}.$$

Since the Frobenius norm is

$$\|A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}\|_F = 7.09538 * 10^{-15}$$

is near zero, it follows that the OMP inverse is an outer inverse. Similar conclusion is valid for the MPO and MPOMP inverses:

$$\|A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}\|_F = 4.70311 * 10^{-15}$$

$$\|A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} - A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)}\|_F = 8.59751 * 10^{-15}.$$

2.7 Summary

The OMP, MPO and MPOMP inverses, i.e. composite outer inverses, are presented in [108] in order to unify and extend the notions of the core, dual core, DMP, MPD, CMP, MPCEP and $*$CEPMP inverses, which are very popular generalized inverses in the recent years. The outer inverse and the Moore-Penrose inverse are used to define composite outer inverses. Various characterizations and representations of composite outer inverses are given from [136]. Using the group, Drazin, core-EP and dual core-EP inverses instead of the outer inverse in presented results, we obtain many characterizations and expressions for the core, dual core, DMP, MPD, CMP, MPCEP and $*$CEPMP inverses. Notice that some of these characterizations and expressions for the core, dual core, DMP, MPD, CMP, MPCEP and $*$CEPMP inverses, are well-known, but some of them are new in literature.

A clear summarization of these particular cases is given in Table 2.1.

Table 2.1: Particular cases of composite outer inverses.

Restrictions	$A^{(2)}_{T,S}$	Result	Ref.
$m=n$, $\mathrm{ind}(A)=1$	$A^{(2)}_{T,S}=A^{\#}$	$A^{(2),\dagger}_{\mathcal{R}(A),\mathcal{N}(A^*)}=A^{\oplus}=A^{\#}AA^{\dagger}$	[2]
$m=n$, $\mathrm{ind}(A)=1$	$A^{(2)}_{T,S}=A^{\#}$	$A^{\dagger,(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}=A_{\oplus}=A^{\dagger}AA^{\#}$	[21]
$m=n$	$A^{(2)}_{T,S}=A^{\mathrm{D}}$	$A^{(2),\dagger}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\dagger})}=A^{\mathrm{D},\dagger}=A^{\mathrm{D}}AA^{\dagger}$	[86]
$m=n$	$A^{(2)}_{T,S}=A^{\mathrm{D}}$	$A^{\dagger,(2)}_{\mathcal{R}(A^{\dagger}A^k),\mathcal{N}(A^k)}=A^{\dagger,\mathrm{D}}=A^{\dagger}AA^{\mathrm{D}}$	[86]
$m=n$	$A^{(2)}_{T,S}=A^{\mathrm{D}}$	$A^{\dagger,(2),\dagger}_{\mathcal{R}(A^{\dagger}A^k),\mathcal{N}(A^kA^{\dagger})}=A^{c,\dagger}=A^{\dagger}AA^{\mathrm{D}}AA^{\dagger}$	[88]
$m=n$	$A^{(2)}_{T,S}=A^{\textcircled{\dagger}}$	$A^{\dagger,(2)}_{\mathcal{R}(A^{\dagger}A^k),\mathcal{N}((A^k)^*)}=A^{\dagger,\textcircled{\dagger}}=A^{\dagger}AA^{\textcircled{\dagger}}$	[19]
$m=n$	$A^{(2)}_{T,S}=A_{\textcircled{\dagger}}$	$A^{\dagger,(2)}_{\mathcal{R}((A^k)^*),\mathcal{N}(A^kA^{\dagger})}=A_{\textcircled{\dagger},\dagger}=A_{\textcircled{\dagger}}AA^{\dagger}$	[19]

One possibility for further research is the generalization to power-composite outer inverses, proposed in [108]. Another possible extension of OMP, MPO and MPOMP inverses could be their extension to tensors case.

Chapter 3

Least squares properties of generalized inverses

3.1 Least squares and best approximate solutions

The Moore-Penrose inverse and certain solutions to some of Penrose equations play fundamental role concerning solutions to the general system of linear equations (or SoLE)

$$Ax = b, \quad A \in \mathbb{C}^{m\times n},\ b \in \mathbb{C}^m \tag{3.1}$$

with respect to unknowns $x \in \mathbb{C}^n$. Fundamental result is restated in Theorem 3.1.

Theorem 3.1.1. *The linear system* (3.1) *is solvable if and only if* $b \in \mathcal{R}(A)$. *Equivalently,* (3.1) *has a solution if and only if* $AA^\dagger b = b$.

In this case, a general solution to (3.1) *is of the form*

$$x = A^\dagger b + (I - A^\dagger A)y, \quad \textit{for arbitrary } y \in \mathbb{C}^n. \tag{3.2}$$

An arbitrary inconsistent SoLE, given by

$$Ax = b, \quad A \in \mathbb{C}^{m\times n},\ b \notin \mathcal{R}(A) \tag{3.3}$$

has no solution. Then the problem is to find an x which minimizes the residual $Ax - b$. Then a vector $u \in \mathbb{C}^n$ is called a *least squares solution* to (3.3) if

$$\|Au - b\| \le \|Ax - b\|, \quad \forall x \in \mathbb{C}^n.$$

The following proposition, restated from [8], shows that $\|Ax-b\|$ is minimized by the vector $x = A^{(1,3)}b$. This statement establishes very important relation between the set of $\{1,3\}$-inverses and the least-squares solutions of the system (3.1).

Proposition 3.1.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $b \in \mathbb{C}^m$. *Then* $\|Ax - b\|$ *is smallest when* $x = A^{(1,3)}b$, *where* $A^{(1,3)} \in A\{1,3\}$. *Conversely, if* $X \in \mathbb{C}^{n\times m}$ *has the property that, for all* b, $\|Ax - b\|$ *is smallest when* $x = Xb$, *then* $X \in A\{1,3\}$.

Since $A^{(1,3)}$ inverse of a matrix is not unique, as a consequence, a SoLE has many least-squares solutions in general. However, among all least-squares solutions of a given SoLE, there exists only one such *solution of minimum norm.*

Definition 3.1.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $b \in \mathbb{C}^m$. *A vector* $\hat{x}$, *which satisfies the minimization problem*

$$\|\hat{x}\| = \min_{x\in\mathbb{C}^n} \|x\|, \quad \text{subject to } Ax = b, \tag{3.4}$$

is called a minimal-norm solution *of the system* $Ax = b$.

The next proposition, restated from [8], establishes a relation between $\{1,4\}$-inverses and the minimum-norm solutions of the linear system $Ax = b$.

Proposition 3.1.2. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $b \in \mathbb{C}^m$. *If* $Ax = b$ *is consistent, the unique solution* x *for which* $\|x\|$ *is smallest is given by* $x = A^{(1,4)}b$, *where* $A^{(1,4)} \in A\{1,4\}$. *Conversely, if* $X \in \mathbb{C}^{n\times m}$ *is such that, whenever* $Ax = b$ *has a solution,* $x = Xb$ *is the solution of minimal-norm, then* $X \in A\{1,4\}$.

The *least-squares solution of minimum norm* is known as *best approximate solution.* Joining the results from Proposition 3.1.1 and Proposition 3.1.2, we are coming to the most important property of the Moore-Penrose inverse.

Corollary 3.1.1. (Penrose 1955) [122] *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then, among the least-squares solutions of $Ax = b$, the solution $A^\dagger b$ is the one of minimum-norm. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that Xb is the* minimal-norm least-squares solution *of $Ax = b$ for all b, then $X = A^\dagger$.*

In the essence, Corollary 3.1.1 shows that $A^\dagger b$ is the minimal-norm least-squares solution of the linear system $Ax = b$. This fact caused a dramatic increase of the interest in the generalized inverses theory.

Furthermore, the next proposition characterizes the set of all least-squares solutions of a given SoLE.

Proposition 3.1.3. (Nashed 1970, 1976) [119, 118] For $A \in \mathbb{C}^{m\times n}$, the set S of all least-squares solutions of the system $Ax = b$ is given by

$$S = A^\dagger b \oplus \mathcal{N}(A) = \{A^\dagger b + (I - A^\dagger A)y \mid \; y \in \mathbb{C}^n\}.$$

These results are extended in solving the linear matrix equations (LME) $AX = B$. More precisely, the Moore-Penrose inverse satisfies the following inequalities [122]:

$$\|AX - B\| \geq \|AA^\dagger B - B\| \tag{3.5}$$

for all X, with equality in (3.5) if and only if

$$X = A^\dagger B + (I - A^\dagger A)L,$$

where L is arbitrary matrix of appropriate dimensions. Moreover,

$$\|A^\dagger B + (I - A^\dagger A)L\| \geq \|A^\dagger B\|, \tag{3.6}$$

with equality in (3.6) if and only if $(I - A^\dagger A)L = 0$.

Penrose's inequalities (3.5) and (3.6) has been extended in [85] to the supremum norm and the L_p norm as well as to the set of $\{1, 3\}$ inverses. This result is restated here for complex matrices.

Proposition 3.1.4. *Let $A \in \mathbb{C}^{m\times n}$ and $A^{(1,3)}$ be an $\{1, 3\}$ inverse of A. Then for all X*

$$\|AX - B\| \geq \|AA^{(1,3)}B - B\|, \tag{3.7}$$

with equality in (3.7) *if and only if*

$$X = A^{(1,3)}B + (I - A^{(1,3)}A)L,$$

where L is arbitrary. Furthermore, the choice $A^{(1,3)} := A^\dagger$ leads to the least squares solution of minimum norm, equal to $A^\dagger B$:

$$\|A^\dagger B + (I - A^\dagger A)L\| \geq \|A^\dagger B\|. \tag{3.8}$$

The properties of the Moore-Penrose inverse presented in this section have caused a real expansion in the study of generalized inverses. It showed the great usability of generalized inverses in solving systems of linear equations as well as matrix equations. In this way, the theory developed for the classical matrix inverse is continued for arbitrary matrices, including square singular as well as rectangular matrices.

3.2 Least-square properties of the Drazin-inverse solution

In the papers [12, 164, 171], the authors present some minimal properties of the Drazin-inverse solution. It can be argued that, in some way, these properties correspond to the properties of the Moore-Penrose inverse solution. Namely, in [12] it is shown that if $b \in \mathcal{R}(A^k)$, where $k = \mathrm{ind}(A)$, then the Drazin-inverse solution is the unique solution of the system $Ax = b$ which belongs to $\mathcal{R}(A^k)$. Also, Wei et al. in [164, 171] proved that the Drazin-inverse solution of the system $Ax = b$ is a solution of minimum P-norm, where P is the matrix included in the Jordan decomposition $A = PJP^{-1}$ of the matrix A.

A major result in this area led to the use of the Drazin inverse in solving some matrix equations and systems of linear equations. More precisely, the utility of the Moore-Penrose inverse in generating least squares solutions $A^*Ax = A^*b$ is extended to linear systems of the form $A^{k+1}x = A^k$, where $k \geq \mathrm{ind}(()A)$.

The obtained results related to the Drazin-inverse solution of a given system SoLE, are inspiration to investigate possibilities if they can be used in order to calculate the Drazin inverse of a given matrix, i.e., to find the Drazin-inverse solution of the matrix equation $AXB = D$, in general. With appropriate modifications, it is possible to find the solution in the form $A^{\mathrm{D}}GB^{\mathrm{D}}$. The matrix $A^{\mathrm{D}}GB^{\mathrm{D}}$ is not always a solution of the matrix equation $AXB = D$, but however it can be always used in order to calculate the Drazin inverse of arbitrary matrix.

The results of this section are complement to the results investigated in [171]. Namely, they are motivated form the idea of defining a gradient iterative method for computing the Drazin-inverse solution of the system (3.9). The goal is achieved by establishing a relation between the Drazin-inverse solution and the linear system (3.9).

Theorem 3.2.1. [166] *Each solution to*

$$Ax = b, \quad b \in \mathcal{R}(A^k), \; k = \mathrm{ind}(A) \tag{3.9}$$

is also a solution to

$$A^{p+1}x = A^p b \qquad p \geq k, \tag{3.10}$$

but the opposite statement does not hold.

Proof. Clearly $Ax = b$, $b \in \mathcal{R}(A^k)$ implies $A^p(Ax - b) = 0$ for $p \geq k$.

On the other hand, Wei in [164] proved that the general solution of (3.9) is given by

$$x = A^{\mathrm{D}}b + A^{k-1}(I - A^{\mathrm{D}}A)z, \tag{3.11}$$

where z is an arbitrary vector. □

The solution $A^{\mathrm{D}}b$ is known as the Drazin-inverse solution of (3.9).

Remark that the opposite statement is not valid, since not every element from $\mathcal{N}(A^k)$ can be represented as $A^{k-1}(I - A^{\mathrm{D}}A)z$, $z \in \mathbb{C}^n$ is arbitrary. Consequently, not every solution of (3.10) is a solution of the equation (3.9) nor a solution of the equation (3.1).

Theorem 3.2.2. [171] *Consider $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{ind}(A)$. The Drazin inverse solution $A^{\mathrm{D}}b$ is the unique solution in $\mathcal{R}(A^k)$ to the system*

$$A^{k+1}x = A^k b. \tag{3.12}$$

Theorem 3.2.3. [171] *Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ and $k = \mathrm{ind}(A)$. The set of all solutions of the equation* (3.12) *is given by*

$$x = A^{\mathrm{D}}b + \mathcal{N}(A^k). \tag{3.13}$$

3.3 Least-square properties of outer inverses

The outer generalized inverses with prescribed range and null-space are very important in matrix theory. The {2}-inverses have application in defining iterative methods for solving the nonlinear equations [8], in statistics [45] as well as in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverses [118].

This section shows that outer inverses with prescribed range and null space are useful in solving the restricted SoLE. This application is based on the following essential result from [22]:

Proposition 3.3.1. [22] *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$, and let the condition*

$$b \in AT, \quad \dim(AT) = \dim(T)$$

be satisfied. Then the unique solution to the constrained SoLE

$$Ax = b, \quad x \in T$$

is given by

$$x = A^{(2)}_{T,S}b,$$

for any subspace S of $\mathbb{C}^m$ satisfying $AT \oplus S = \mathbb{C}^m$.

Further investigations show that some new classes of generalized inverses are applicable in solving corresponding unconstrained and constrained SoLE. These generalized inverses are composed of appropriate outer inverses and the Moore-Penrose inverse and surveyed in the subsequent Section 2.3.

3.4 Least squares properties of composite inverses

The goal of this section is to show usability of arbitrary generalized inverses in solving SoLE and matrix equations. Practically, each kind of generalized inverses is related to appropriate matrix equation and/or linear system. In this section we will show that the composite outer inverses inherit and generalize least squares properties of the Moore-Penrose inverse as well as the minimal properties that the Drazin inverse and the outer inverses exhibit in solving unconstrained and constraines SoLE. In order to find appropriate approximations to inconsistent SoLE $Ax = b$, $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^n$ and $b \in \mathbb{C}^m$, one typical approach is to asks for, so called, *generalized solutions*, defined as solutions to $GAx = Gb$ with respect to an appropriate matrix $G \in \mathbb{C}^{n\times m}$ [84]. It is important to mention that the system $GAx = Gb$ is consistent in the case $\mathrm{rank}(GA) = \mathrm{rank}(G)$. Such approach has been exploited extensively. One particular choice is $G = A^*$, which leads to widely used least-squares solutions obtained as solutions to the *normal equation* $A^*Ax = A^*b$. Another important choice is $m = n$, $G = A^k$ and $k = \mathrm{ind}(A)$, which leads to the so called Drazin normal equation (3.12) and usage of the Drazin inverse solution $A^{\mathrm{D}}b$. The set of all solutions to $A^{k+1}x = A^kb$ is given by $x = A^{\mathrm{D}}b + \mathcal{N}(A^k)$ [168]. Moreover, $x = A^{\mathrm{D}}b$ is the unique solution to $A^{k+1}x = A^kb$ on the set $\mathcal{R}(A^k)$ [168]. The SoLE of the form

$$GAx = Gb, \quad A \in \mathbb{R}_r^{m\times n},\ G \in \mathbb{R}_s^{n\times m},\ 0 < s \le r \tag{3.14}$$

was considered in [139]. The $m \times n$ matrix A in (3.14) is given and $n \times m$ matrix G is chosen such that $\mathrm{rank}(GA) = \mathrm{rank}(G) = s \le r$.

Theorem 3.4.1. [139] *The set of all solutions of the SoLE* (3.14) *under the assumptions* $\mathrm{rank}(GA) = \mathrm{rank}(G)$ *is given by*

$$x = A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}\, b + \mathcal{N}(GA). \tag{3.15}$$

The particular case $G = A^k$, $k = \mathrm{ind}(A)$ of (3.15) was investigated in [**?**].

The *least squares properties of composite inverses* will be used in the sense that the general solution to the system $G(Ax = b)$ is given in the form

$$x = Xb + (I - A^{(2)}_{T,S}A)y, \tag{3.16}$$

or in the form

$$x = Xb + (I - XA)y, \tag{3.17}$$

where X is one of composite outer inverses considered in Section 2.3.

3.4.1 Least squares properties of the DMP inverse

The choice $G := AA^\dagger = P_{\mathcal{R}(A)}$ gives the projection $P_{\mathcal{R}(A)}b$ of b on $\mathcal{R}(A)$. Then the induced SoLE

$$GAx = Gb \Longleftrightarrow Ax = P_{\mathcal{R}(A)}b, \quad \mathrm{ind}(A) = k \tag{3.18}$$

is consistent. Then the Drazin-inverse solution to (3.18) is equal to

$$x = A^{\mathrm{D}}(P_{\mathcal{R}(A)}b) = (A^{\mathrm{D}}AA^\dagger)b = A^{\mathrm{D},\dagger}b,$$

which is just the DMP-inverse solution to $Ax = b$.

As it was shown in [12], $A^{\mathrm{D}}b$ is a solution of the following system

$$Ax = b, \quad \text{where } b \in \mathcal{R}(A^k),\ k = \mathrm{ind}(A). \tag{3.19}$$

In the case $b \in \mathcal{R}(A^k)$, the general solution to $Ax = b$, is

$$x = A^{\mathrm{D},\dagger}b + (I - A^{\mathrm{D}}A)y = A^{\mathrm{D}}b + (I - A^{\mathrm{D}}A)y = A^{\mathrm{D},\dagger}b + (I - A^{\mathrm{D},\dagger}A)y, \quad y \in \mathcal{R}(A^k).$$

Theorem 3.4.2. [79] *Let* (2.1) *be the Schur decomposition of* $A \in \mathbb{C}^{n\times n}$ *of index* $\mathrm{ind}(A) = k$. *Further, assume that the columns of* V *and* U^* *are the bases for* $\mathcal{N}(A^k)$ *and* $\mathcal{N}(A^{k^*})$, *respectively. Consider the matrix*

$$E = V(UV)^{-1}U.$$

Then the range and the kernel of E *are equal to* $\mathcal{R}(E) = \mathcal{R}(V) = \mathcal{N}(A^k)$ *and* $\mathcal{N}(E) = \mathcal{N}(U) = \mathcal{R}(A^k)$. *Furthermore,* $A^k + E$ *is a nonsingular matrix, its inverse is*

$$(A^k + E)^{-1} = (A^k)^\# + E^\#,$$

and the DMP *inverse solution* $A^{\mathrm{D},\dagger}b$ *of* $Ax = b$ *satisfies*

$$A^{\mathrm{D},\dagger}b = (A^k + E)^{-1}AA^\dagger b.$$

3.4.2 Least squares properties of the MPCEP-inverse

Solvability of certain systems of linear equations in terms of expressions involving the MPCEP inverse was investigated in [107].

Corollary 3.4.1. [107] *If $A \in \mathbb{C}^{n\times n}$ of index $\operatorname{ind}(A) = k$ and $b \in \mathbb{C}^n$, the SoLE*

$$Ax = A^k(A^k)^\dagger b \tag{3.20}$$

is consistent and its general solution is

$$x = A^{\dagger,\textcircled{\dagger}}b + (I_n - A^\dagger A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. The results follows from [108, Theorem 3.1] in the particular case $A^{(2)}_{T,S} = A^{\textcircled{\dagger}}$. □

In the specific case when $b \in \mathcal{R}(A^k)$ in Corollary 3.4.1, it is possible to derive the following consequence.

Corollary 3.4.2. [107] *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, the constrained SoLE (3.9) is consistent and its general solution is*

$$x = A^{\dagger,\textcircled{\dagger}}b + (I_n - A^\dagger A)y = A^\dagger b + (I_n - A^\dagger A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Since $b \in \mathcal{R}(A^k)$ gives $b = A^k(A^k)^\dagger b$, one can deduce $A^{\dagger,\textcircled{\dagger}}b = A^\dagger A^k(A^k)^\dagger b = A^\dagger b$. The rest of the proof follows from Corollary 3.4.1. □

Theorem 3.4.3. [107] *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, $A^{\dagger,\textcircled{\dagger}}b$ is the unique solution to (3.20) in $\mathcal{R}(A^\dagger A^k)$.*

Proof. By Corollary 3.4.1we conclude that $A^{\dagger,\textcircled{\dagger}}b$ is a solution in $\mathcal{R}(A^\dagger A^k)$ of (3.20).

For two solutions $x, x_1 \in \mathcal{R}(A^\dagger A^k)$ of (3.20), notice that

$$x - x_1 \in \mathcal{R}(A^\dagger A^k) \cap \mathcal{N}(A) \subseteq \mathcal{R}(A^{\dagger,\textcircled{\dagger}}A) \cap \mathcal{N}(A^{\dagger,\textcircled{\dagger}}A) = \{0\}.$$

Hence, $x = x_1$ and so the unique solution to (3.20) is $x = A^{\dagger,\textcircled{\dagger}}b$. □

3.4.3 Least squares properties of the OMP and MPO inverses

In this section we investigate possibility to apply the composite inverses in solving an inconsistent SoLE (3.1). One possible projection $G(Ax = b)$ is defined by $G = AA^\dagger(= P_{\mathcal{R}(A)})$, which projects $Ax = b$ on the consistent SoLE (3.18). As a consequence, the $A^{(2)}_{T,S}$ solution to (3.18) is equal to

$$x = A^{(2)}_{T,S}\left(P_{\mathcal{R}(A)}b\right) = \left(A^{(2)}_{T,S}AA^\dagger\right)b = A^{(2),\dagger}_{T,S}\, b,$$

which coincides with the OMP-inverse solution to $Ax = b$. So, an important conclusion is that the OMP-inverse solution to inconsistent system $Ax = b$ coincides with the the $A^{(2)}_{T,S}$ solution to the projected consistent system (3.18).

Applying the MPO inverse, some systems of linear equations were solved in [108].

Theorem 3.4.4. [108] *For $A \in \mathbb{C}^{m\times n}_{T,S}$, the SoLE*

$$Ax = AA^{(2)}_{T,S}b \tag{3.21}$$

is consistent and its general solution is equal to

$$x = A^{\dagger,(2)}_{T,S}b + (I - A^\dagger A)y, \tag{3.22}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. For x given by (3.22), we get

$$Ax = AA^{\dagger,(2)}_{T,S}b = AA^\dagger AA^{(2)}_{T,S}b = AA^{(2)}_{T,S}b$$

and so x is a solution to (3.21).

If x is a solution to (3.21), we have

$$A^{\dagger,(2)}_{T,S}b = A^\dagger\left(AA^{(2)}_{T,S}b\right) = A^\dagger Ax.$$

Thus,

$$x = A^{\dagger,(2)}_{T,S}\, b + x - A^\dagger Ax = A^{\dagger,(2)}_{T,S}b + (I - A^\dagger A)x,$$

i.e., the solution x is of the form (3.22). □

Remark 3.4.1. *The proof of Corollary 3.4.1 is derived using* [108, Theorem 3.1]. *It is important to mention that the proof can be derived using Theorem 3.4.4.*

Under an extra assumption $b \in \mathcal{R}(AA^{(2)}_{T,S})$ in Theorem 3.4.4, we get the following consequence.

Corollary 3.4.3. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the SoLE*

$$Ax = b, \qquad b \in \mathcal{R}(AA^{(2)}_{T,S})$$

is consistent and its general solution is of the form (3.22).

3.4.4 Least squares properties of the weak group inverse

The results of this section are mainly based on the results from [116]. The weak group inverse can be applied in solving appropriate SoLE.

Theorem 3.4.5. [116] *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, the equation*

$$\left(A^{k+2}\right)^* A^2 x = \left(A^{k+2}\right)^* Ab, \qquad b \in \mathbb{C}^n, \tag{3.23}$$

is consistent and its general solution is

$$x = A^{ⓦ} b + (I - A^{ⓦ} A)y, \tag{3.24}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Suppose that x is represented as in (3.24). Applying $A^{ⓦ} = A^k(A^{k+2})^\dagger A$, we have $(A^{k+2})^* A^2 A^{ⓦ} = (A^{k+2})^* A$. Therefore, $(A^{k+2})^* A^2 x = (A^{k+2})^* Ab$, which implies that (3.23) is true for x defined as in (3.24).

For a solution x to (3.23), one obtains

$$\begin{aligned}
A^{ⓦ} b &= A^k \left(A^{k+2}\right)^\dagger Ab = A^k (A^{k+2})^\dagger \left(\left(A^{k+2}\right)^\dagger\right)^* \left(A^{k+2}\right)^* Ab \\
&= A^k \left(A^{k+2}\right)^\dagger \left(\left(A^{k+2}\right)^\dagger\right)^* \left(A^{k+2}\right)^* A^2 x = A^k \left(A^{k+2}\right)^\dagger A^2 x \\
&= A^{ⓦ} Ax.
\end{aligned}$$

Now, we get

$$x = A^{ⓦ} b + x - A^{ⓦ} Ax = A^{ⓦ} b + (I - A^{ⓦ} A)x,$$

i.e., x possesses the form (3.24). $\square$

3.4.5 Least squares properties of the core–EP inverse

The results of Corollary 3.4.4 can be concluded using Theorem 3.4.1 in conjunction with stated representations of composite outer inverses.

Corollary 3.4.4. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$.*

(a) *The set of all solutions of the SoLE* (3.14) *under the choice $G := A^k\left(A^k\right)^*$ (i.e. $A^k\left(A^k\right)^* Ax = A^k\left(A^k\right)^* b$), is given by*

$$\begin{aligned}
x &= A^{(2)}_{\mathcal{R}(A^k(A^k)^*),\mathcal{N}(A^k(A^k)^*)} b + \mathcal{N}\left(A^k\left(A^k\right)^* A\right) \\
&= A^{\textcircled{\dagger}} b + \mathcal{N}\left(\left(A^k\right)^* A\right).
\end{aligned} \tag{3.25}$$

(b) *The set of all solutions of the SoLE* (3.14) *in the case $G := A^k A^\dagger$ (i.e. $A^k x = A^k A^\dagger b$), is given by*

$$\begin{aligned}
x &= A^{(2)}_{\mathcal{R}\left(A^k A^\dagger\right),\mathcal{N}\left(A^k A^\dagger\right)} b + \mathcal{N}\left(A^k\right) \\
&= A^{\mathrm{D},\dagger} b + \mathcal{N}\left(A^k\right).
\end{aligned} \tag{3.26}$$

(c) *The set of all solutions of the SoLE* (3.14) *in the case $G := A^\dagger A^k\left(A^k\right)^*$ (i.e. $A^\dagger A^k\left(A^k\right)^* Ax = A^\dagger A^k\left(A^k\right)^* b$) is given by*

$$\begin{aligned}
x &= A^{(2)}_{\mathcal{R}\left(A^\dagger A^k\left(A^k\right)^*\right),\mathcal{N}\left(A^\dagger A^k\left(A^k\right)^*\right)} b + \mathcal{N}\left(A^\dagger A^k\left(A^k\right)^* A\right) \\
&= A^{\dagger,\textcircled{\dagger}} b + \mathcal{N}\left(\left(A^k\right)^* A\right).
\end{aligned} \tag{3.27}$$

(d) *The set of all solutions of the SoLE* (3.14) *in the case* $G := A^\dagger A^k A^\dagger$ (*i.e.* $A^\dagger A^k x = A^\dagger A^k A^\dagger b$), *is given by*

$$\begin{aligned} x &= A^{(2)}_{\mathcal{R}\left(A^\dagger A^k A^\dagger\right),\mathcal{N}\left(A^\dagger A^{2k} A^\dagger\right)}\, b + \mathcal{N}\left(A^\dagger A^k\right) \\ &= A^{c,\dagger} b + \mathcal{N}\left(A^k\right). \end{aligned} \tag{3.28}$$

3.5 Summary

Least squares solutions have achieved great importance in solving inconsistent SoLE. Minimum norm least square solution is closest to the zero and unique. It is termed as the best approximate solution. One important application of the Moore-Penrose inverse is a presentation of the best approximation solution of a SoLE or LME. Some characterizations of least square solutions and the solution of minimum norm are considered. Beside the best approximation solution, the Drazin-inverse solution and the outer-inverse solution exhibit some attracted many attention in literature. Elementary properties of the Drazin-inverse solution and the outer-inverse solution are presented. Applicability of composite generalized inverses in solving unconstrained or constrained systems of linear equations is systematized and investigated based on [136]. Least squares properties of composite outer inverses are collected.

Further research could be oriented towards the investigation of further composite outer inverses and their minimal and least squares properties. Also, applications of composite outer inverses in solving appropriate constrained optimization problems is an interesting problems.

The data included in Table 3.1 summarizes results presented in this chapter.

Table 3.1: Particular cases of composite outer inverse solutions.

No	Restrictions	Equation/Model	Solution	Ref.
1.	$\mathrm{rank}(GA) = \mathrm{rank}(G)$	$GAx = Gb$	$x = A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}\, b + \mathcal{N}(GA)$	[139]
2.	$m = n$, $\mathrm{ind}(A) = k$	$Ax = P_{\mathcal{R}(A)} b$	$x = A^{\mathrm{D}}(P_{\mathcal{R}(A)} b) = A^{\mathrm{D},\dagger} b$	[79]
3.	$A \in \mathbb{C}^{m\times n}_{T,S}$	$Ax = P_{\mathcal{R}(A)} b$	$x = A^{(2)}_{T,S}(P_{\mathcal{R}(A)} b) = A^{(2),\dagger} b$	[108]
4.	$m = n$, $\mathrm{ind}(A) = k$	$Ax = A^k (A^k)^\dagger b$	$x = A^{\dagger,Ⓣ} b + (I_n - AA^\dagger) y$	[107]
5.	$m = n$, $\mathrm{ind}(A) = k$	$Ax = b$, $b \in \mathcal{R}(A^k)$	$x = A^{\dagger,Ⓣ} b + (I_n - AA^\dagger) y$ $= A^\dagger b + (I_n - AA^\dagger) y$	[107]
6.	$A \in \mathbb{C}^{m\times n}_{T,S}$	$Ax = AA^{(2)}_{T,S} b$	$x = A^{\dagger,(2)}_{T,S} b + (I_n - AA^\dagger) y$	[108]
7.	$m = n$, $\mathrm{ind}(A) = k$	$\left(A^{k+2}\right)^* A^2 x = \left(A^{k+2}\right)^* Ab$	$x = A^{ⓦ} b + (I - A^{ⓦ} A) y$	[116]

Chapter 4

Solvability of matrix approximation problems

For a nonsingular matrix $M \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$, the unique solution of $Mx = b$ is $x = M^{-1}b$. Let $M(j \to b)$ be a matrix obtained from M replacing the jth column of M by b. The Cramer's rule for the solution $x = (x_1, x_2, \ldots, x_n)^\top$ of the nonsingular system of equations $Mx = b$ is [5, 127]

$$x_j = \frac{\det(M(j \to b))}{\det(M)}, \quad j = 1, \ldots, n. \tag{4.1}$$

In a special case $M \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(M) = k$ and $b \in \mathcal{R}(M^k)$, Wang [155] presented a Cramer's rule for the unique Drazin inverse solution $x = M^{\mathrm{D}}b$ to the restricted linear equation [12]

$$Mx = b, \qquad x \in \mathcal{R}(M^k).$$

When $M \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = \mathrm{ind}(MW)$, $k_2 = \mathrm{ind}(WM)$ and $b \in \mathcal{R}((WM)^{k_2})$, Wei [162] gave a Cramer's rule for the W-weighted Drazin inverse solution $x = M^{\mathrm{D},W}b$ to the general restricted linear equation

$$WMWx = b, \qquad x \in \mathcal{R}((MW)^{k_1}).$$

In particular, for $M \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(M) = 1$ and $b \in \mathcal{R}(M)$, $x = M^{\#}b$ presents the unique solution to $Mx = b$. Notice that $M^{\#}b = M^{\oplus}b$ and Cramer's rule for finding the solution $x = M^{\oplus}b$ was proposed in [80]. Without the assumption $b \in \mathcal{R}(M)$, for $M \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(M) = 1$ and $b \in \mathbb{C}^n$, Wang and Zhang in [159] provided Cramer's rule for finding $x = M^{\oplus}b$ to the next constrained matrix approximation problem, stated in the Frobenius norm as follows:

$$\min \|Mx - b\|_F \quad \text{subject to} \quad x \in \mathcal{R}(M).$$

Using the core-EP inverse, the first goal of this chapter is to obtain the unique solution to the following more-general constrained matrix minimization problem in the Euclidean norm [111]:

$$\min \|Mx - b\|_2 \quad \text{subject to} \quad x \in \mathcal{R}(M^k), \tag{4.2}$$

where $b \in \mathbb{C}^n$, $M \in \mathbb{C}^{n\times n}$ and $k = \mathrm{Ind}(M)$. Thus, we solve the problem which generalizes the problem proposed in [159] for complex matrices with index one to complex matrices with arbitrary index. Observe that assumption $b \in \mathcal{R}(M^k)$, which appeared in [12, 155], is omitted.

We present two kinds of Cramer's rules to find the unique solution to (4.2) based on one well-known expression and one novel representation for the core-EP inverse.

We also propose a solution to the next constrained problem [111]:

$$\min \|WMWx - b\|_2 \quad \text{subject to} \qquad x \in \mathcal{R}((MW)^k), \tag{4.3}$$

where $W \in \mathbb{C}^{n\times m}$ is a nonzero matrix, $M \in \mathbb{C}^{m\times n}$, $k = \max\{\mathrm{Ind}(MW), \mathrm{Ind}(WM)\}$ and $b \in \mathbb{C}^n$.

Extension of these results to the class of quaternion matrices can be found in [67, 68].

4.1 Solvability of (4.2) based on core-EP inverse

Throughout this section, it is supposed that $M \in \mathbb{C}^{n\times n}$ and $k = \mathrm{Ind}(M)$. The following decomposition of matrix M is well-known, and the expression for core-EP inverse of M was verified in [158].

Lemma 4.1.1. [158] *There exist a unitary matrix $P \in \mathbb{C}^{n\times n}$, a nonsingular matrix $M_1 \in \mathbb{C}^{t\times t}$, $t = \operatorname{rank}(M^k)$, and a nilpotent matrix $M_3 \in \mathbb{C}^{(n-t)\times(n-t)}$ of index k with*

$$M = P \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} P^*. \tag{4.4}$$

In addition,

$$M^{\textcircled{\dagger}} = P \begin{bmatrix} M_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^*. \tag{4.5}$$

We also state a useful relation between a certain invertible bordered matrix and core-EP inverse, proved in [81].

Lemma 4.1.2. [81, Theorem 2.3] *Let two full column rank matrices U^* and V satisfy*

$$\mathcal{N}(U) = \mathcal{R}(M^k) \qquad \text{and} \qquad \mathcal{R}(V) = \mathcal{N}\left((M^k)^*\right). \tag{4.6}$$

Then the bordered matrix

$$S = \begin{bmatrix} M & V \\ U & 0 \end{bmatrix}$$

is nonsingular and its inverse is equal to

$$S^{-1} = \begin{bmatrix} M^{\textcircled{\dagger}} & (I - M^{\textcircled{\dagger}} M)U^\dagger \\ V^\dagger (I - MM^{\textcircled{\dagger}}) & -V^\dagger (I - MM^{\textcircled{\dagger}})MU^\dagger \end{bmatrix}.$$

We now obtain the unique solution of the constrained matrix approximation problem (4.2), using the core-EP inverse. Note that Theorem 4.1.1 incudes [159, Theorem 3.1] as a particular case.

Theorem 4.1.1. [111] *The unique solution to* (4.2) *is*

$$x = M^{\textcircled{\dagger}} b.$$

Proof. Because of $x \in \mathcal{R}(M^k)$, there exists $y \in \mathbb{C}^n$ such that $x = M^k y$. Let M be represented as in (4.4),

$$P^* y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad P^* b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad y_1, b_1 \in \mathbb{C}^t.$$

By (4.5), we get

$$M^{\textcircled{\dagger}} b = P \begin{bmatrix} M_1^{-1} b_1 \\ 0 \end{bmatrix}.$$

For a corresponding matrix $Y \in \mathbb{C}^{t\times(n-t)}$, we obtain

$$\begin{aligned} Mx &= M^{k+1} y = P \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} \begin{bmatrix} M_1^k & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= P \begin{bmatrix} M_1^{k+1} & M_1 Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= P \begin{bmatrix} M_1^{k+1} y_1 + M_1 Y y_2 \\ 0 \end{bmatrix} \end{aligned}$$

and thus

$$\begin{aligned} \|Mx - b\|_2^2 &= \left\| \begin{bmatrix} M_1^{k+1} y_1 + M_1 Y y_2 - b_1 \\ -b_2 \end{bmatrix} \right\|_2^2 \\ &= \|M_1^{k+1} y_1 + M_1 Y y_2 - b_1\|_2^2 + \|b_2\|_2^2. \end{aligned}$$

Since x is a solution to (4.2) if and only if y is a solution to $\|M^{k+1} y - b\|_2 = \min$, notice that $\min\limits_{y_2, y_2} \|M_1^{k+1} y_1 + M_1 Y y_2 - b_1\|_2^2 = 0$, i.e. $\|M^{k+1} y - b\|_2 = \min = \|b_2\|_2$ for arbitrary y_2 and

$$y_1 = M_1^{-(k+1)} b_1 - M_1^{-k} Y y_2.$$

Thus,

$$\begin{aligned} x &= M^k y = P \begin{bmatrix} M_1^k & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P \begin{bmatrix} M_1^k y_1 + Y y_2 \\ 0 \end{bmatrix} \\ &= P \begin{bmatrix} M_1^{-1} b_1 - Y y_2 + Y y_2 \\ 0 \end{bmatrix} = P \begin{bmatrix} M_1^{-1} b_1 \\ 0 \end{bmatrix} \\ &= M^{\textcircled{\dagger}} b \end{aligned}$$

is the unique solution to (4.2). $\square$

Firstly, we present the Cramer's rule related to finding unique solution to (4.2) based on Lemma 4.1.2.

Theorem 4.1.2. [111] *Let two full column rank matrices U^* and V satisfy* (4.6)*. The unique solution $x = (x_1, x_2, \dots, x_n)^\top$ to* (4.2) *can be represented as*

$$x_j = \det\left(\begin{bmatrix} M(j \to b) & V \\ U(j \to 0) & 0 \end{bmatrix}\right) / \det\left(\begin{bmatrix} M & V \\ U & 0 \end{bmatrix}\right), \quad j = 1, \dots, n. \tag{4.7}$$

Proof. From $x = M^{\textcircled{\dagger}} b \in \mathcal{R}(M^k) = \mathcal{N}(U)$, we have $Ux = 0$. Now, the solution of (4.2) satisfies

$$\begin{bmatrix} M & V \\ U & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

By Lemma 4.1.2 and the Cramer rule (4.1), $x = M^{\textcircled{\dagger}} b$ and its componentwise representation (4.7) follow. □

We now establish new representations for the core-EP inverse. Notice that Theorem 4.1.3 generalizes corresponding representation of the core inverse proved in [159, Theorem 3.5]. The following notations will be used for the sake of simplicity:

$$P := M^k (M^k)^*, \quad Q := M^k (M^k)^* M + VV^* = PM + VV^*. \tag{4.8}$$

Theorem 4.1.3. [111] *Let a matrix V satisfy*

$$\mathcal{N}(V^*) = \mathcal{R}(M^k) \quad (\textit{or equivalently } \mathcal{R}(V) = \mathcal{N}((M^k)^*)) \tag{4.9}$$

and P, Q are defined as in (4.8)*. Then Q is nonsingular and*

$$M^{\textcircled{\dagger}} = Q^{-1} P. \tag{4.10}$$

Proof. For $T = M^k (M^k)^* M$, we firstly observe that $\mathcal{R}(T) \subseteq \mathcal{R}(M^k)$. Further, by

$$\mathcal{R}(M^k) = \mathcal{R}(M^k (M^k)^*) = \mathcal{R}(M^k (M^k)^* M^k) \subseteq \mathcal{R}(M^k (M^k)^* M) = \mathcal{R}(T),$$

we deduce that $\mathcal{R}(T) = \mathcal{R}(M^k)$. Therefore, $\operatorname{rank}(T) = \operatorname{rank}(M^k) \leq 1$ and $T^{\textcircled{\#}}$ exists.

From $\mathcal{N}(V^*) = \mathcal{R}(M^k) = \mathcal{R}(M^{\textcircled{\dagger}})$, we obtain $V^* M^{\textcircled{\dagger}} = 0$ and

$$V^* T^{\textcircled{\#}} = V^* T (T^{\textcircled{\#}})^2 = V^* M^k (M^k)^* M (T^{\textcircled{\#}})^2 = 0.$$

Let $Y = T^{\textcircled{\#}} + (VV^*)^\dagger - M^{\textcircled{\dagger}} M (VV^*)^\dagger$. Then, we get

$$TM^{\textcircled{\dagger}} = M^k (M^k)^* M M^{\mathrm{D}} M^k (M^k)^\dagger = M^k (M^k)^* M^k (M^k)^\dagger = M^k (M^k)^*.$$

Hence,

$$\begin{aligned} (T + VV^*) Y &= TT^{\textcircled{\#}} + T(VV^*)^\dagger - T(VV^*)^\dagger + VV^*(VV^*)^\dagger \\ &= TT^{\textcircled{\#}} + VV^*(VV^*)^\dagger = P_{\mathcal{R}(T)} + P_{\mathcal{R}(VV^*)} \\ &= P_{\mathcal{R}(M^k)} + P_{\mathcal{R}(V)} = P_{\mathcal{N}(V^*)} + P_{\mathcal{N}(V^*)^\perp} \\ &= I \end{aligned}$$

implies that $T + VV^*$ is nonsingular. Since $(T + VV^*) M^{\textcircled{\dagger}} = M^k (M^k)^*$, notice that

$$M^{\textcircled{\dagger}} = (T + VV^*)^{-1} M^k (M^k)^*,$$

i.e., (4.10) holds. □

Similarly as in Theorem 4.1.3, we verify the second representation for the core-EP inverse under the same assumption (4.9).

Theorem 4.1.4. [111] *Let a matrix V satisfy* (4.9)*. Then $M^{k+1} + VV^*$ is nonsingular and*

$$M^{\textcircled{\dagger}} = M^k \left(M^{k+1} + VV^*\right)^{-1}. \tag{4.11}$$

Proof. Since $\operatorname{rank}(M^{k+1}) = \operatorname{rank}(M^k) \leq 1$, we conclude that $(M^{k+1})^{\textcircled{\#}}$ exists. For

$$Y = (M^{k+1})^{\textcircled{\#}} + (VV^*)^\dagger - M^{\mathrm{D}} M (VV^*)^\dagger,$$

we obtain

$$\begin{aligned} (M^{k+1} + VV^*) Y &= M^{k+1} (M^{k+1})^{\textcircled{\#}} + VV^* (VV^*)^\dagger \\ &= P_{\mathcal{R}(M^k)} + P_{\mathcal{R}(V)} = P_{\mathcal{N}(V^*)} + P_{\mathcal{N}(V^*)^\perp} \\ &= I \end{aligned}$$

and so $M^{k+1} + VV^*$ is nonsingular. The equality $M^{\textcircled{\dagger}} \left(M^{k+1} + VV^*\right) = M^{k+1}$ gives that (4.11) is satisfied. □

By the proofs of Theorem 4.1.3 and Theorem 4.1.4, remark that expressions (4.10) and (4.11) hold if we replace k with arbitrary $l \geq k$.

In the case when $k = 1$ in Theorem 4.1.4, we can obtain new representation of the core inverse.

Corollary 4.1.1. *Let a matrix V satisfy* (4.9). *Then $M^2 + VV^*$ is nonsingular and*

$$M^{\#\!\!\!\!\bigcirc} = M\left(M^2 + VV^*\right)^{-1}.$$

Applying Theorem 4.1.3, we can propose the following Cramer's rule for finding the solution to (4.2).

Theorem 4.1.5. *Let a matrix V satisfy* (4.9) *and P, Q are defined as in* (4.8). *Then the unique solution $x = (x_1, x_2, \ldots, x_n)^\top$ to* (4.2) *satisfies*

$$x_j = \frac{\det\left(Q(j \to P\,b)\right)}{\det(Q)}, \quad j = 1, \ldots, n. \tag{4.12}$$

Proof. By Theorem 4.1.1 and Theorem 4.1.3,

$$x = M^{\textcircled{\dagger}} b = Q^{-1} P\, b.$$

So,

$$Qx = P\,b.$$

The rest follows by the Cramer rule (4.1). □

As usual, e_j represents a jth column of the identity matrix. Inspired by the condensed determinantal expressions for Moore-Penrose and Drazin inverses presented in [53] and for the core inverse given in [159], we proved the condensed determinantal expression for the core-EP inverse.

Theorem 4.1.6. *Let a matrix V satisfy* (4.9). *The core-EP inverse $M^{\textcircled{\dagger}}$ is represented as*

$$M^{\textcircled{\dagger}}_{j,l} = \frac{\det\left(Q(j \to P\,e_l)\right)}{\det(Q)}, \quad j, l = 1, \ldots, n. \tag{4.13}$$

Proof. Using

$$Qx = P\,e_l, \;\; l = 1, \ldots, n$$

we obtain

$$e_j^\top x = \frac{\det\left(Q(j \to P\,e_l)\right)}{\det(Q)}, \;\; j, l = 1, \ldots, n.$$

Applying Theorem 4.1.3 and $M^{\textcircled{\dagger}}_{j,l} = e_j^\top M^{\textcircled{\dagger}} e_l$, we finish the proof. □

4.2 Solvability of (4.3) based on weighted core-EP inverse

In this section, it is supposed that $W \in \mathbb{C}^{n\times m}$ is a nonzero matrix, $M \in \mathbb{C}^{m\times n}$ and $k = \max\{\mathrm{Ind}(MW), \mathrm{Ind}(WM)\}$. Moreover, the following extensions of the notations (4.8) will be used:

$$\begin{aligned} P_W &:= M(WM)^k[(WM)^{k+1}]^*, \\ Q_W &:= M(WM)^k[(WM)^{k+1}]^* WMW + U^*U = P_W WMW + U^*U. \end{aligned} \tag{4.14}$$

Lemma 4.2.1. [170][40, Theorem 4.1 and Theorem 5.2] *There exist unitary matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$, two nonsingular matrices $M_1, W_1 \in \mathbb{C}^{t\times t}$, and matrices $M_3 \in \mathbb{C}^{(m-t)\times(n-t)}$, $W_3 \in \mathbb{C}^{(n-t)\times(m-t)}$ such that M_3W_3 and W_3M_3 are nilpotent of indices* $\mathrm{ind}(MW)$ *and* $\mathrm{ind}(WM)$, *respectively, with*

$$M = P\begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} Q^* \quad \text{and} \quad W = Q\begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix} P^*. \tag{4.15}$$

In addition,

$$M^{\textcircled{\dagger},W} = P\begin{bmatrix} (W_1M_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^*. \tag{4.16}$$

In [78], Ma generalized Lemma 4.1.2 related to an invertible bordered matrix and core-EP inverse to certain invertible bordered matrix and weighted core-EP inverse.

Lemma 4.2.2. [78, Theorem 2.2] *Let two full column rank matrices U^* and V satisfy*

$$\mathcal{N}(U) = \mathcal{R}((MW)^k) \qquad \text{and} \qquad \mathcal{R}(V) = \mathcal{N}\left(((WM)^k)^*\right). \tag{4.17}$$

Then the bordered matrix

$$S = \begin{bmatrix} WMW & V \\ U & 0 \end{bmatrix}$$

is nonsingular and

$$S^{-1} = \begin{bmatrix} M^{\textcircled{\dagger},W} & (I - M^{\textcircled{\dagger},W}WMW)U^\dagger \\ V^\dagger(I - WMWM^{\textcircled{\dagger},W}) & -V^\dagger(I - WMWM^{\textcircled{\dagger},W})WMWU^\dagger \end{bmatrix}. \tag{4.18}$$

Theorem 4.2.1 confirms that the constrained matrix approximation problem (4.3) can be solved by means of the W-weighted core-EP inverse.

Theorem 4.2.1. [111] *The unique solution to* (4.3) *is*

$$x = M^{\textcircled{\dagger},W}b.$$

Proof. By the hypothesis $x \in \mathcal{R}((MW)^k)$, there exists $y \in \mathbb{C}^m$ such that $x = (MW)^k y$. Assume that M and W are given by (4.15),

$$P^*y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad Q^*b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad y_1, b_1 \in \mathbb{C}^t.$$

Using (4.16), note that

$$M^{\textcircled{\dagger},W}b = P\begin{bmatrix} (W_1M_1W_1)^{-1}b_1 \\ 0 \end{bmatrix}.$$

Because, for $E = M_1W_2 + M_2W_3$ and a corresponding matrix Y,

$$\begin{aligned} WMWx &= WMW(MW)^k y \\ &= Q\begin{bmatrix} W_1M_1W_1 & W_1E + W_2M_3W_3 \\ 0 & W_3M_3W_3 \end{bmatrix}\begin{bmatrix} M_1W_1 & E \\ 0 & M_3W_3 \end{bmatrix}^k\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= Q\begin{bmatrix} W_1M_1W_1 & W_1E + W_2M_3W_3 \\ 0 & W_3M_3W_3 \end{bmatrix}\begin{bmatrix} (M_1W_1)^k & Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= Q\begin{bmatrix} W_1(M_1W_1)^{k+1} & W_1M_1W_1Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= Q\begin{bmatrix} W_1(M_1W_1)^{k+1}y_1 + W_1M_1W_1Yy_2 \\ 0 \end{bmatrix}, \end{aligned}$$

we obtain

$$\begin{aligned} \|WMWx - b\|_2^2 &= \left\| \begin{bmatrix} W_1(M_1W_1)^{k+1}y_1 + W_1M_1W_1Yy_2 - b_1 \\ -b_2 \end{bmatrix} \right\|_2^2 \\ &= \|W_1(M_1W_1)^{k+1}y_1 + W_1M_1W_1Yy_2 - b_1\|_2^2 + \|b_2\|_2^2. \end{aligned}$$

We now observe that $\min\limits_{y_2,y_2} \|W_1(M_1W_1)^{k+1}y_1 + W_1M_1W_1Yy_2 - b_1\|_2^2 = 0$ for

$$y_1 = (M_1W_1)^{-(k+1)}W_1^{-1}b_1 - (M_1W_1)^{-k}Yy_2.$$

Hence,

$$\begin{aligned} x &= (MW)^k y = P\begin{bmatrix} (M_1W_1)^k & Y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= P\begin{bmatrix} (M_1W_1)^k y_1 + Yy_2 \\ 0 \end{bmatrix} \\ &= P\begin{bmatrix} (W_1M_1W_1)^{-1}b_1 - Yy_2 + Yy_2 \\ 0 \end{bmatrix} \\ &= P\begin{bmatrix} (W_1M_1W_1)^{-1}b_1 \\ 0 \end{bmatrix} \\ &= M^{\textcircled{\dagger},W}b \end{aligned}$$

is the unique solution to (4.3). ☐

A Cramer's rule representation of the solution to (4.3) is given as an application of Lemma 4.2.2.

Theorem 4.2.2. [111] *Let two full column rank matrices U^* and V satisfy* (4.17)*. Then the unique solution $x = M^{\textcircled{\dagger},W} b$ to* (4.3) *can be expressed componentwise by*

$$x_j = \det\left(\begin{bmatrix} WMW(j \to b) & V \\ U(j \to 0) & 0 \end{bmatrix}\right) / \det\left(\begin{bmatrix} WMW & V \\ U & 0 \end{bmatrix}\right), \quad j = 1, \dots, n.$$

Proof. Because $x = M^{\textcircled{\dagger},W} b \in \mathcal{R}((MW)^k) = \mathcal{N}(U)$, we get $Ux = 0$. Hence, the solution of (4.3) satisfies

$$\begin{bmatrix} WMW & V \\ U & 0 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Applying Lemma 4.2.2, $x = M^{\textcircled{\dagger},W} b$ and, by the Cramer rule, corresponding elementwise statement follows. □

We can present new expressions for W-weighted core-EP inverse. The first expression is proved in the case when $\mathcal{N}(U) = \mathcal{R}((MW)^k)$.

Theorem 4.2.3. [111] *Let a matrix U^* satisfy*

$$\mathcal{N}(U) = \mathcal{R}((MW)^k) \tag{4.19}$$

and P_W, Q_W are defined as in (4.14)*. Then Q_W is nonsingular and*

$$M^{\textcircled{\dagger},W} = Q_W^{-1} P_W. \tag{4.20}$$

Proof. Let $T = M(WM)^k[(WM)^{k+1}]^* WMW$. Since

$$T^* = (WMW)^*(WM)^{k+1}[(MW)^k M]^* = (WMW)^*(WM)^{k+1} M^*[(MW)^k]^*,$$

we observe that $\mathcal{N}([(MW)^k]^*) \subseteq \mathcal{N}(T^*)$. Then, by

$$\begin{aligned}
\mathcal{N}([(MW)^k]^*) &= \mathcal{N}([(MW)^{k+1}]^*) = \mathcal{N}(W^*[(WM)^k]^* M^*) \supseteq \mathcal{N}([(WM)^k]^* M^*) \\
&= \mathcal{N}([(WM)^k]^\dagger (WM)^k [(WM)^k]^* M^*) \\
&= \mathcal{N}([(WM)^{k+1}]^\dagger (WM)^{k+1}[(WM)^k]^* M^*) \supseteq \mathcal{N}((WM)^{k+1}[(WM)^k]^* M^*) \\
&= \mathcal{N}(([(WM)^{k+1}]^\dagger)^*[(WM)^{k+1}]^*(WM)^{k+1}[(WM)^k]^* M^*) \\
&\supseteq \mathcal{N}((WMW)^*(WM)^{k+1}[(MW)^k M]^*) = \mathcal{N}(T^*),
\end{aligned}$$

we deduce that $\mathcal{N}(T^*) = \mathcal{N}([(MW)^k]^*)$. Thus, $\mathcal{R}(T) = \mathcal{R}((MW)^k)$ which yields $\operatorname{rank}(T) = \operatorname{rank}((MW)^k) \le 1$ and $T^{\textcircled{\#}}$ exists. The hypothesis $\mathcal{N}(U) = \mathcal{R}((MW)^k) = \mathcal{R}(M^{\textcircled{\dagger},W})$ implies $UM^{\textcircled{\dagger},W} = 0$ and

$$UT^{\textcircled{\#}} = UT(T^{\textcircled{\#}})^2 = U(MW)^k M[(WM)^{k+1}]^* WMW(T^{\textcircled{\#}})^2 = 0.$$

Set $Y = T^{\textcircled{\#}} + (U^*U)^\dagger - M^{\textcircled{\dagger},W} WMW(U^*U)^\dagger$. Notice that

$$TM^{\textcircled{\dagger},W} = M(WM)^k[(WM)^{k+1}]^*(WM)^k[(WM)^k]^\dagger = M(WM)^k[(WM)^{k+1}]^*.$$

Therefore,

$$\begin{aligned}
(T + U^*U)Y &= TT^{\textcircled{\#}} + T(U^*U)^\dagger - T(U^*U)^\dagger + U^*U(U^*U)^\dagger \\
&= TT^{\textcircled{\#}} + U^*U(U^*U)^\dagger = P_{\mathcal{R}(T)} + P_{\mathcal{R}(U^*U)} \\
&= P_{\mathcal{R}((MW)^k)} + P_{\mathcal{R}(U^*)} = P_{\mathcal{R}((MW)^k)} + P_{\mathcal{R}((MW)^k)^\perp} \\
&= I,
\end{aligned}$$

which indicates that $T + U^*U$ is nonsingular. From $(T + U^*U)M^{\textcircled{\dagger},W} = P$, we conclude that is satisfied. □

Theorem 4.2.4 gives the second representation for the W-weighted core-EP inverse under the assumption $\mathcal{R}(V) = \mathcal{N}(((WM)^k)^*)$.

Theorem 4.2.4. [111] *Let a matrix V satisfy*

$$\mathcal{R}(V) = \mathcal{N}(((WM)^k)^*). \tag{4.21}$$

Then $(WM)^{k+2} + VV^*$ is nonsingular and

$$M^{\textcircled{\dagger},W} = M(WM)^k \left((WM)^{k+2} + VV^*\right)^{-1}. \tag{4.22}$$

Proof. Because $\mathrm{rank}((WM)^{k+2}) = \mathrm{rank}((WM)^k) \leq 1$, then $((WM)^{k+2})^{\#}$ exists. Set $Y = ((WM)^{k+2})^{\#} + (VV^*)^{\dagger} - (WM)^{\mathrm{D}} WM (VV^*)^{\dagger}$. Then

$$\begin{aligned} ((WM)^{k+2} + VV^*)Y &= (WM)^{k+2}((WM)^{k+2})^{\#} + VV^*(VV^*)^{\dagger} \\ &= P_{\mathcal{R}((WM)^k)} + P_{\mathcal{R}(V)} = P_{\mathcal{N}(V^*)} + P_{\mathcal{N}(V^*)^{\perp}} \\ &= I, \end{aligned}$$

that is, $(WM)^{k+2} + VV^*$ is nonsingular. Thus, $M^{\textcircled{\dagger},W}\left((WM)^{k+2} + VV^*\right) = M(WM)^k$ yields $M^{\textcircled{\dagger},W} = M(WM)^k\left((WM)^{k+2} + VV^*\right)^{-1}$. $\square$

Using Theorem 4.2.3, one more Cramer's rule for unique solution of (4.3) is obtained.

Theorem 4.2.5. [111] *Let a matrix U satisfy* (4.19). *Then the unique solution $x = (x_1, \ldots, x_m)^{\top}$ to* (4.3) *satisfies*

$$x_j = \frac{\det\left(Q_W(j \to P_W\, b)\right)}{\det(Q_W)},\ j = 1, \ldots, m.$$

Proof. According to Theorem 4.2.1 and Theorem 4.2.3, it follows

$$x = M^{\textcircled{\dagger},W} b = Q_W^{-1} P_W\, b,$$

which implies

$$Q_W\, x = P_W\, b.$$

Using the Cramer rule, the proof is completed. $\square$

Based on Theorem 4.2.3, for the W-weighted core-EP inverse, we get the condensed determinantal expression.

Theorem 4.2.6. [111] *Let a matrix U satisfy* (4.19). *Then the W-weighted core-EP inverse $M^{\textcircled{\dagger},W}$ is elementwise represented by*

$$M^{\textcircled{\dagger},W}_{j,l} = \frac{\det\left(Q_W(j \to P_W\, e_l)\right)}{\det(Q_W)},$$

where $j = 1, \ldots, m, l = 1, \ldots, n$.

Proof. By

$$W_W\, x = P_W\, e_l,\ \ l = 1, \ldots, n$$

for $j = 1, \ldots, m, l = 1, \ldots, n$, we have

$$e_j^{\top} x = \frac{\det\left(Q_W(j \to P_W\, e_l)\right)}{\det(Q)},\ j = 1, \ldots, m, l = 1, \ldots, n.$$

The proof follows by Theorem 4.2.3 and $M^{\textcircled{\dagger},W}_{j,l} = e_j^{\top} M^{\textcircled{\dagger},W} e_l$. $\square$

4.3 Numerical verification

Example 4.3.1. This example is a verification of Theorem 4.1.2 on the input data given by

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad b = \{0., 0., 0.127546, 0.099768\}^{\top} \in \mathcal{R}(M^2).$$

Matrices

$$U = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

satisfy (4.6). The bordered matrix S is equal to

$$S = \begin{bmatrix} M & V \\ U & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 \\ 2 & 4 & 6 & 8 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$S_1 = \begin{bmatrix} M(1 \to b) & V \\ U(1 \to 0) & 0 \end{bmatrix} = \begin{bmatrix} 0. & 1 & 0 & 0 & 1 & 0 \\ 0. & 0 & 0 & 0 & 0 & 1 \\ 0.127546 & 0 & 0 & 1 & 0 & 0 \\ 0.099768 & 0 & 1 & 0 & 0 & 0 \\ 0. & 3 & 5 & 7 & 0 & 0 \\ 0. & 4 & 6 & 8 & 0 & 0 \end{bmatrix}.$$

Then $x_1 = \det(S_1)/\det(S) = 0.35.486$. Continuing in the same way, it is possible to verify that (4.7) generates first four elements from the vector

$$S^{-1}b = \{0.35486, -0.582174, 0.099768, 0.127546, 0.582174, 0.\}^\top.$$

Indeed,

$$S_2 = \begin{bmatrix} M(2 \to b) & V \\ U(2 \to 0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0. & 0 & 0 & 1 & 0 \\ 0 & 0. & 0 & 0 & 0 & 1 \\ 0 & 0.127546 & 0 & 1 & 0 & 0 \\ 0 & 0.099768 & 1 & 0 & 0 & 0 \\ 1 & 0. & 5 & 7 & 0 & 0 \\ 2 & 0. & 6 & 8 & 0 & 0 \end{bmatrix},$$

$$x_2 = \det(S_2)/\det(S) = -0.582174;$$

$$S_3 = \begin{bmatrix} M(3 \to b) & V \\ U(3 \to 0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0. & 0 & 1 & 0 \\ 0 & 0 & 0. & 0 & 0 & 1 \\ 0 & 0 & 0.127546 & 1 & 0 & 0 \\ 0 & 0 & 0.099768 & 0 & 0 & 0 \\ 1 & 3 & 0. & 7 & 0 & 0 \\ 2 & 4 & 0. & 8 & 0 & 0 \end{bmatrix},$$

$$x_3 = \det(S_3)/\det(S) = 0.099768;$$

$$S_4 = \begin{bmatrix} M(4 \to b) & V \\ U(4 \to 0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0. & 1 & 0 \\ 0 & 0 & 0 & 0. & 0 & 1 \\ 0 & 0 & 0 & 0.127546 & 0 & 0 \\ 0 & 0 & 1 & 0.099768 & 0 & 0 \\ 1 & 3 & 5 & 0. & 0 & 0 \\ 2 & 4 & 6 & 0. & 0 & 0 \end{bmatrix},$$

$$x_4 = \det(S_4)/\det(S) = 0.127546.$$

Example 4.3.2. Observe the matrix

$$M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & -1 & 2 \end{bmatrix}$$

of index $k = \operatorname{ind}(M) = 2$. The core-EP inverse of M is equal to

$$M^{\textcircled{\dagger}} = M^{\mathrm{D}} M^k (M^k)^\dagger = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{12} & -\frac{1}{12} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{12} & \frac{1}{12} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}. \tag{4.23}$$

It can be verified that

$$N = \mathcal{N}\left((M^2)^{\mathrm{T}}\right) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $V = N^{\mathrm{T}}$ satisfies (4.9). Application of (4.10) and (4.11) gives

$$\left(M^k (M^k)^{\mathrm{T}} M + VV^{\mathrm{T}}\right)^{-1} M^k (M^k)^{\mathrm{T}} = M^{\mathrm{D}} M^k (M^k)^\dagger = M^k \left(M^{k+1} + VV^{\mathrm{T}}\right)^{-1}.$$

Example 4.3.3. For the matrix M in Example 4.3.2 it can be verified

$$Q := M^k(M^k)^{\mathrm{T}}M + VV^{\mathrm{T}} = \begin{bmatrix} 17 & -15 & 0 & 0 & 0 & 0 \\ -15 & 17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27 & -25 & -30 & 30 \\ 0 & 0 & -25 & 27 & 30 & -30 \\ -2 & -2 & -74 & 72 & 160 & -158 \\ -2 & -2 & 72 & -74 & -158 & 160 \end{bmatrix},$$

$$P := M^k(M^k)^{\mathrm{T}} = \begin{bmatrix} 17 & -15 & 0 & 0 & 0 & 0 \\ -15 & 17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27 & -25 & -30 & 30 \\ 0 & 0 & -25 & 27 & 30 & -30 \\ -2 & -2 & -74 & 72 & 160 & -158 \\ -2 & -2 & 72 & -74 & -158 & 160 \end{bmatrix}.$$

and $\det(Q) = 1990656$. Further, $M^k(M^k)^{\mathrm{T}}e_1 = \{8, -8, 0, 0, 0, 0\}^{\mathrm{T}}$, implies

$$\det\left(Q(1 \to P\,e_1)\right) = \det\left(\begin{bmatrix} 8 & -15 & 0 & 0 & 0 & 0 \\ -8 & 17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27 & -25 & -30 & 30 \\ 0 & 0 & -25 & 27 & 30 & -30 \\ 0 & -2 & -74 & 72 & 160 & -158 \\ 0 & -2 & 72 & -74 & -158 & 160 \end{bmatrix}\right) = 497664.$$

So, according to (4.13), $M^{\textcircled{\dagger}}_{j,l} = 497664/1990656 = 1/4$, which is in accordance with (4.23).

Further, consider the vector

$$b = \{0.132264, -0.132264, 0.0358987, -0.0358987, 0.893685, 0.57147\}^{\mathrm{T}} \in \mathcal{R}(M^2).$$

Then

$$Q(1 \to P\,b) = \begin{bmatrix} 2.11622 & -15 & 0 & 0 & 0 & 0 \\ -2.11622 & 17 & 0 & 0 & 0 & 0 \\ -2.64775 & 0 & 27 & -25 & -30 & 30 \\ 2.64775 & 0 & -25 & 27 & 30 & -30 \\ 17.8245 & -2 & -74 & 72 & 160 & -158 \\ -14.8942 & -2 & 72 & -74 & -158 & 160 \end{bmatrix}$$

and

$$x_1 = \frac{\det\left(Q(1 \to P\,b)\right)}{\det(Q)} = 131646/1990656 = 0.0661319.$$

Similar computation gives

$$Q(2 \to P\,b) = \begin{bmatrix} 17 & 2.11622 & 0 & 0 & 0 & 0 \\ -15 & -2.11622 & 0 & 0 & 0 & 0 \\ 0 & -2.64775 & 27 & -25 & -30 & 30 \\ 0 & 2.64775 & -25 & 27 & 30 & -30 \\ -2 & 17.8245 & -74 & 72 & 160 & -158 \\ -2 & -14.8942 & 72 & -74 & -158 & 160 \end{bmatrix}$$

and

$$x_2 = \frac{\det\left(Q(2 \to P\,b)\right)}{\det(Q)} = -131646/1990656 = -0.0661319.$$

Simple verification approves the equalities $x_1 = (M^{\textcircled{\dagger}}b)_1, x_2 = (M^{\textcircled{\dagger}}b)_2$. Continuing in the same way, the vector

$$x = M^{\textcircled{\dagger}}b = \{0.0661319, -0.0661319, 0.0179494, -0.0179494, 0.792264, 0.672893\}^{\top}$$

can be generated using (4.12), as

$$\left\{\frac{\det\left(Q(j \to P\,b)\right)}{\det(Q)}, j = 1, \dots, 6\right\}.$$

Example 4.3.4. Consider the matrices

$$M = \begin{bmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

with $k = \max\{\mathrm{ind}(MW), \mathrm{ind}(WM)\} = 1$. Then, for

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

it holds (4.19) and the matrix

$$Q_W = M(WM)^k[(WM)^{k+1}]^{\mathrm{T}}WMW + U^{\mathrm{T}}U = \begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix} \tag{4.24}$$

is nonsingular. Further, consider the matrix

$$P_W := M(WM)^k[(WM)^{k+1}]^{\mathrm{T}} = \begin{bmatrix} 6 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 2 \end{bmatrix}. \tag{4.25}$$

In accordance with (4.20), the W-weighted core-EP inverse of M shall be equal to

$$M^{\oplus,W} = Q_W^{-1} P_W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}. \tag{4.26}$$

On the other hand, the matrix

$$V = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

satisfies (4.21). Then,

$$(WM)^{k+2} + VV^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 1 \end{bmatrix}$$

is nonsingular and according to (4.22), the value (4.26) of $M^{\oplus,W}$ can be generated again, i.e.,

$$M^{\oplus,W} = M(WM)^k \left((WM)^{k+2} + VV^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}.$$

Example 4.3.5. Consider the matrices M, W and U of Example 4.3.4 and the vector $b = \{-6, -1, 0, -6\}^{\mathrm{T}} \in \mathcal{R}((WM)^k)$. The expressions Q and P are defined as in (4.24) and (4.25), respectively. Further calculation gives the list

$$\begin{aligned} &\{Q_W(1 \to P_W\, b),\ Q_W(2 \to P_W\, b),\ Q_W(3 \to P_W\, b)\} \\ &= \left\{ \begin{bmatrix} -74 & -12 & 12 \\ 0 & 3 & 0 \\ -29 & -4 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -12 & -74 \\ 0 & 3 & 0 \\ 5 & -4 & -29 \end{bmatrix} \right\}. \end{aligned}$$

We invoke Theorem 4.2.5, which implies

$$\begin{aligned} \{x_1, x_2, x_3\} &= \left\{ \frac{\det\left(Q_W(1 \to P_W\, e_1)\right)}{\det(Q_W)}, \frac{\det\left(Q_W(2 \to P_W\, e_1)\right)}{\det(Q_W)}, \frac{\det\left(Q_W(3 \to P\, e_1)\right)}{\det(Q_W)} \right\} \\ &= \frac{\det\left(\begin{bmatrix} -74 & -12 & 12 \\ 0 & 3 & 0 \\ -29 & -4 & 4 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix}\right)}, \frac{\det\left(\begin{bmatrix} 2 & -74 & 12 \\ 0 & 0 & 0 \\ 5 & -29 & 4 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix}\right)}, \frac{\det\left(\begin{bmatrix} 2 & -12 & -74 \\ 0 & 3 & 0 \\ 5 & -4 & -29 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix}\right)}. \\ &= \{-1, 0, -6\}. \end{aligned}$$

Finally, this numerical experience is in accordance with the solution vector $x = M^{\oplus,W}b$ is $x = \{-1, 0, -6\}^{\mathrm{T}}$.

The validity of Theorem 4.2.6 can be easily verified. Let's actually test the element $M^{\textcircled{\dagger},W}_{3,1}$, for which we know from example 4.3.4 that it is equal to $\frac{1}{2}$. Then, from Theorem 4.2.6 we have

$$M^{\textcircled{\dagger},W}_{3,1} = \frac{\det\left(Q_W(3 \to P_W\, e_1)\right)}{\det(Q_W)} = \frac{\det\left(\begin{bmatrix} 2 & -12 & 6 \\ 0 & 3 & 0 \\ 5 & -4 & 2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix}\right)} = \frac{1}{2}.$$

Likewise, we can verify all elements of the matrix $M^{\textcircled{\dagger},W}$ in (4.26). Indeed,

$$\{Q_W(j \to P_W\, e_l), \{j,3\}, \{l,4\}\}$$
$$= \begin{bmatrix} \begin{bmatrix} 6 & -12 & 12 \\ 0 & 3 & 0 \\ 2 & -4 & 4 \end{bmatrix} & \begin{bmatrix} 2 & -12 & 12 \\ 0 & 3 & 0 \\ 5 & -4 & 4 \end{bmatrix} & \begin{bmatrix} 0 & -12 & 12 \\ 0 & 3 & 0 \\ 0 & -4 & 4 \end{bmatrix} & \begin{bmatrix} 6 & -12 & 12 \\ 0 & 3 & 0 \\ 2 & -4 & 4 \end{bmatrix} \\ \begin{bmatrix} 2 & 6 & 12 \\ 0 & 0 & 0 \\ 5 & 2 & 4 \end{bmatrix} & \begin{bmatrix} 2 & 2 & 12 \\ 0 & 0 & 0 \\ 5 & 5 & 4 \end{bmatrix} & \begin{bmatrix} 2 & 0 & 12 \\ 0 & 0 & 0 \\ 5 & 0 & 4 \end{bmatrix} & \begin{bmatrix} 2 & 6 & 12 \\ 0 & 0 & 0 \\ 5 & 2 & 4 \end{bmatrix} \\ \begin{bmatrix} 2 & -12 & 6 \\ 0 & 3 & 0 \\ 5 & -4 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -12 & 2 \\ 0 & 3 & 0 \\ 5 & -4 & 5 \end{bmatrix} & \begin{bmatrix} 2 & -12 & 0 \\ 0 & 3 & 0 \\ 5 & -4 & 0 \end{bmatrix} & \begin{bmatrix} 2 & -12 & 6 \\ 0 & 3 & 0 \\ 5 & -4 & 2 \end{bmatrix} \end{bmatrix},$$

which generates the matrix

$$\frac{1}{\det(Q_W)} \begin{bmatrix} 0 & -156 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -78 & 0 & 0 & -78 \end{bmatrix}$$

identical with $M^{\textcircled{\dagger},W}$ in (4.26).

4.4 Comparison of the core-EP inverse solution with LS solutions

In this section, the core-EP inverse solution is compared with two solutions obtained by two corresponding LS problems. The first one is the LS problem constrained by linear equality constraints (known as LSE) and given in the form of the minimization problem

$$\min_{Bx=d} \|Mx - b\|_2, \tag{4.27}$$

where $M \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^m$, $d \in \mathbb{R}^p$, $\mathrm{rank}(B) = p < n$. An approximate solution (4.27) is based on the LS problem without constraints

$$\min_{x\in\mathbb{R}^n} \left\| \begin{bmatrix} M \\ \lambda B \end{bmatrix} x - \begin{bmatrix} b \\ \lambda d \end{bmatrix} \right\|_2, \tag{4.28}$$

where $\lambda \gg 1$ is a large real number [77, 181]. How can we exploit the LSE problem? Our objective is the minimization (4.2), which could be derived from (4.27) in the choice

$$m = p = n,\ B = I_n,\ x = d = M^k y,\ y \in \mathbb{R}^n,\ k = \mathrm{ind}(M).$$

So, we tend to find the solution to

$$\min_{y\in\mathbb{R}^n} \left\| \begin{bmatrix} M \\ \lambda I \end{bmatrix} M^k y - \begin{bmatrix} b \\ \lambda M^k y \end{bmatrix} \right\|_2. \tag{4.29}$$

Then, the following numerical experiments can be performed:
Compute the solution y to (4.29) and then compare the vector $x = M^k y$ with $x_1 = M^{\textcircled{\dagger}} b$ by $\|x - x_1\|$. Also, it will be useful to compare $\|Mx - b\|_2$ with $\|Mx_1 - b\|_2$.

The second LS approach used for comparison is based on the observation that the equality-constrained LS problem (4.2) is equivalent to the following unconstrained LS problem:

$$\min_{y\in\mathbb{R}^n} \|M^{k+1} y - b\|_2. \tag{4.30}$$

Then, we can perform the following numerical experiments:
Generate the solution y to (4.30) and compare $x = M^k y$ with $x_1 = M^{\textcircled{\dagger}} b$ by the vector norm $\|x - x_1\|$. Also, compare $\|My - b\|_2$ with $\|Mx_1 - b\|_2$.

Example 4.4.1. *In this example, six large-scale randomly generated matrices are considered in numerical experiments with the models* (4.29) *and* (4.30). *In particular, we considered the matrices* $M = M_k = rand(k, k-1) * rand(k-1, k)$, *for* $k = 20, 30$, $M = M_{50} = rand(50, 19) * rand(19, 50)$ *and* $M = M_p = rand(p, p/2) * rand(p/2, p)$, *for* $p = 100, 200, 500$. *The value of* λ *in* (4.29) *was set to* $\lambda = 10^6$ *and the value of the vector* b, *for all cases, was set to*

$$b = 0.001 * ones(size(M_j, 1), 1), \ j = 20, 30, 50, 100, 200, 500.$$

In order to solve the unconstrained optimization problems (4.29) *and* (4.30), *we invoked the MATLAB function* `fminunc` *which starts at the point*

$$x_0 = zeros(size(M_j, 1), 1), \ j = 20, 30, 50, 100, 200, 500.$$

Also, for the core-EP inverse, we have used the formula (see [42]*)*

$$M^{\textcircled{\dagger}} = M^{\mathrm{D}} M^k (M^k)^{\dagger},$$

while the Drazin inverse was computed according to Corollary 2.1 from [140].
The results of Table 4.1 were constructed by finding the solution to the optimization problem (4.29). *Then, using the solution* y *of* (4.29) *we compare the solution* $x = M^k y$ *with the core-EP inverse solution* $x_1 = M^{\textcircled{\dagger}} b$ *by evaluating the norm* $\|x - x_1\|$. *Also, by comparing norm* $\|Mx - b\|_2$ *with* $\|Mx_1 - b\|_2$ *we may conclude that* $\|Mx_1 - b\|_2$ *is smaller in all test cases, so that the core-EP inverse solution is a competitive alternative. The results of Table 4.2 were*

	$\|x - x_1\|_2$	$\|Mx - b\|$	$\|Mx_1 - b\|$
M_{20}	0.0029	6.4465e-04	2.5623e-04
M_{30}	0.0035	0.0013	5.6952e-05
M_{50}	7.7391e-04	0.0020	8.2942e-04
M_{100}	8.4365e-04	0.0012	5.8856e-04
M_{200}	4.3465e-04	2.9260e-10	7.4416e-08
M_{500}	5.3429e-04	0.0224	5.6874e-04

Table 4.1: Numerical experiments for solving (4.29)

constructed by finding the solution to the optimization problem (4.30). *Then, using the solution* y *of* (4.30) *we compare the solution* $x = M^k y$ *with the core-EP inverse solution* $x_1 = M^{\textcircled{\dagger}} b$ *by evaluating the norm* $\|x - x_1\|$. *Again, we may conclude that the core-EP inverse solution is a competitive alternative, since* $\|Mx_1 - b\|_2$ *is smaller in all test cases with respect to* $\|Mx - b\|_2$.

	$\|x - x_1\|_2$	$\|Mx - b\|$	$\|Mx_1 - b\|$
M_{20}	0.0029	0.0044	2.5623e-04
M_{30}	0.0035	0.0054	5.6952e-05
M_{50}	7.7518e-04	0.0700	8.2942e-04
M_{100}	8.4361e-04	0.0100	5.8856e-04
M_{200}	5.8990e-04	0.0141	7.4416e-08
M_{500}	5.3428e-04	0.0224	5.6874e-04

Table 4.2: Numerical experiments for solving (4.30)

4.5 Applications of considered minimization problems

In the first application we will show that the constrained minimization problem (4.2) covers solutions of some constrained linear systems.

One important case of (4.2), i.e., of (4.30) is the further restriction $b \in \mathcal{R}(M^k)$ in (4.2). Such restriction leads to the following LS problem:

$$\min_y \|M^{k+1} y - M^k c\|_2, \ c \in \mathbb{R}^n. \tag{4.31}$$

Exact solutions of (4.31) are solutions of the Drazin normal equation

$$M^{k+1}y = M^k c. \tag{4.32}$$

Let us introduce an additional assumption $b \in \mathcal{R}(M^k)$ in Theorem 4.1.1. The unique solution of (4.30) is $y = M^{\textcircled{\dagger}}b$. Further, assumption $b \in \mathcal{R}(M^k)$ implies $b = M^k c$, for some $c \in \mathbb{R}^n$, and gives the unique solution od the form

$$y = M^{\textcircled{\dagger}}M^k c = M^{\mathrm{D}}M^k(M^k)^{\dagger}M^k c = M^{\mathrm{D}}M^k c = M^{\mathrm{D}}b.$$

In this way, Proposition 4.5.1 reveals known result from [168].

Proposition 4.5.1. [168] *Let $M \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$ and $k = \mathrm{ind}(M)$. The set of all solutions of the equation* (4.32) *is given by*

$$y = M^{\mathrm{D}}c + \mathcal{N}(M^k),\ \ k = \mathrm{ind}(M). \tag{4.33}$$

Moreover, $y = M^{\mathrm{D}}b$ is the unique solution to (4.32) *satisfying $b \in \mathcal{R}(M^k)$.*

Another application of (4.2) is in the representation of some restricted matrix equations. On the basis of Theorem 4.1.1, it is possible to express solutions to the following more-general constrained matrix minimization problem:

$$\min \|MX - B\|_2 \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}(M^k), \tag{4.34}$$

where $B \in \mathbb{C}^{n\times n}$, $M \in \mathbb{C}^{n\times n}$ and $k = \mathrm{Ind}(M)$.

Corollary 4.5.1. *The unique solution to* (4.34) *is*

$$X = M^{\textcircled{\dagger}}B.$$

Further, assumption $\mathcal{R}(B) \subseteq \mathcal{R}(M^k)$ implies $B = M^k C$, $C \in \mathbb{C}^{n\times n}$ and gives the unique solution to (4.34) in the form

$$X = M^{\textcircled{\dagger}}M^k C = M^{\mathrm{D}}M^k(M^k)^{\dagger}M^k C = M^{\mathrm{D}}M^k C = M^{\mathrm{D}}B.$$

In this way, we just derive known result from [157].

Proposition 4.5.2. [157] *Let $M \in \mathbb{R}^{n\times n}$ of index $k = \mathrm{ind}(M)$, $B \in \mathbb{R}^n$ satisfy $\mathcal{R}(B) \subseteq \mathcal{R}(M^k)$. Then the solution of*

$$AX = B,\ \ \mathcal{R}(X) \subseteq \mathcal{R}(M^k) \tag{4.35}$$

is equal to $X = M^{\mathrm{D}}B$.

Consequently, application of the core-EP inverse in solving constrained matrix equations generalizes known application of the Drazin inverse in solving constrained matrix equations.

Chapter 5

Generalizations of composite inverses

The intention of this chapter is to generalize composite outer inverses and their extension to arbitrary matrices. We were guided by two main ideas. First, the existence of composite outer inverses is limited by strict constraints on the ranks of the input matrices. So, domain of their applicability is reduced. Second, it is clear that the proposed extensions of composite outer inverses will initiate analogous generalizations of known generalized inverses included in the OMP, MPO and MPOMP classes, such as the core, core-EP and *core-EP inverses, the DMP and MPD inverses as well as the CMP, MPCEP and *CEPMP inverses.

5.1 Generalization of core-EP inverse for rectangular matrices

Since composite outer inverses do not involve the core-EP inverse, continuing previous research about composite outer inverses, the goal of this section is to present generalization of the core-EP inverse to rectangular matrices, using appropriate composition of outer inverse and the Moore-Penorse inverse.

Precisely, we define an extension of the core-EP inverse (termed as the g-core-EP inverse) for a rectangular matrix in terms of the Moore-Penrose inverse of a corresponding matrix and the outer inverse $A^{(2)}_{T,S}$ [113]. The core-EP inverse [123] and the core inverse [2] are special cases of the g-core-EP inverse for $A^{(2)}_{T,S} = A^{\mathrm{D}}$ and $A^{(2)}_{T,S} = A^{\#}$, respectively. Also, the Moore-Penrose inverse can be derived after certain choice of the outer inverse in definition of the g-core-EP inverse. As a consequence, this approach defines a wider class of outer inverses.

As the dual of the g-core-EP inverse, the *g-core-EP inverse for a rectangular matrix is presented in the second research stream. Several characterizations and main properties of the g-core-EP and the *g-core-EP inverse are discovered. Integral and limit representations of the g-core-EP and *g-core-EP inverses are developed.

5.1.1 Characterizations of g-core-EP inverse

An algebraic approach enables us to present a new outer inverse applicable to arbitrary matrices.

Theorem 5.1.1. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the system of matrix equations*

$$XAX = X, \quad XA = A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}A \quad \text{and} \quad AX = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} \tag{5.1}$$

possesses the unique solution $X := A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$.

Proof. We can easily show that (5.1) holds for $X := A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$.

If different matrices X and X_1 satisfy the system of matrix equations (5.1), then

$$AX_1 = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = AX, \quad X_1A = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A = XA,$$

which further implies

$$X = (XA)X = X_1(AX) = X_1AX_1 = X_1.$$

Thus, the solution X to the system (5.1) is unique. $\square$

Definition 5.1.1. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The* generalized core-EP (or g-core-EP) inverse *of given matrix A is defined as*

$$A^{\textcircled{2}}_{T,S} := A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}.$$

Various important special cases of the defined g-core-EP inverse, which recover some popular outer inverses, are given now.

(i) For $m = n$, $k = \mathrm{ind}(A)$ and $A^{(2)}_{T,S} = A^{\mathrm{D}}$, the g-core-EP inverse becomes the core-EP inverse. Indeed,

$$\begin{aligned} A^{\textcircled{2}}_{\mathcal{R}(A^k),N(A^k)} &= A^{\mathrm{D}}(AA^{\mathrm{D}})^{\dagger} = A^{\mathrm{D}}AA^{\mathrm{D}}(AA^{\mathrm{D}})^{\dagger} = A^{\mathrm{D}}P_{\mathcal{R}(AA^{\mathrm{D}})} \\ &= A^{\mathrm{D}}P_{\mathcal{R}(A^k)} = A^{\mathrm{D}}A^k(A^k)^{\dagger} = A^{\textcircled{\dagger}}. \end{aligned}$$

(ii) When $k = 1$ in part (i), we have $A^{(2)}_{T,S} = A^{\#}$ and the g-core-EP inverse is equal to the core inverse, i.e., $A^{\textcircled{2}}_{\mathcal{R}(A),N(A)} = A^{\textcircled{\#}}$,.

(iii) If $A^{(2)}_{T,S} = A^{\dagger}$, then $A^{\textcircled{2}}_{\mathcal{R}(A^*),N(A^*)} = A^{\dagger}$.

Corollary 5.1.1 gives a representation of the g-core-EP inverse which is based on corresponding orthogonal projection.

Corollary 5.1.1. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then*

$$A^{\textcircled{2}}_{T,S} = A^{(2)}_{T,S}P_{\mathcal{R}(AA^{(2)}_{T,S})}.$$

Proof. The proof follows from $P_{\mathcal{R}(AA^{(2)}_{T,S})} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$. $\square$

In the next result, we consider projections determined by the g-core-EP inverse and observe that the g-core-EP inverse is an outer inverse with corresponding null space and range.

Lemma 5.1.1. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then:*

(i) *$A^{\textcircled{2}}_{T,S}A$ is a projector onto T along $\mathcal{N}((A^{(2)}_{T,S}A)^*A)$;*

(ii) *$AA^{\textcircled{2}}_{T,S}$ is the orthogonal projector onto $\mathcal{R}(AA^{(2)}_{T,S})$;*

(iii) *$A^{\textcircled{2}}_{T,S} = A^{(2)}_{T,\mathcal{N}((AA^{(2)}_{T,S})^*)}$.*

Proof. (i) Since $A^{\textcircled{2}}_{T,S}$ is an outer inverse of A, we have that $A^{\textcircled{2}}_{T,S}A$ is a projector. Notice that

$$\begin{aligned} \mathcal{R}(A^{\textcircled{2}}_{T,S}A) &\subseteq \mathcal{R}(A^{(2)}_{T,S}) = \mathcal{R}(A^{(2)}_{T,S}AA^{(2)}_{T,S}) \\ &= \mathcal{R}(A^{(2)}_{T,S}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}AA^{(2)}_{T,S}) \\ &\subseteq \mathcal{R}(A^{\textcircled{2}}_{T,S}A), \end{aligned}$$

which gives $\mathcal{R}(A^{\textcircled{2}}_{T,S}A) = \mathcal{R}(A^{(2)}_{T,S}) = T$. Also, we obtain

$$\mathcal{N}(A^{\textcircled{2}}_{T,S}A) = \mathcal{N}\left(\left(AA^{(2)}_{T,S}\right)^{\dagger}A\right) = \mathcal{N}\left(\left(AA^{(2)}_{T,S}\right)^{*}A\right).$$

(ii) By Theorem 5.1.1, $AA^{\textcircled{2}}_{T,S} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ is the orthogonal projector onto $\mathcal{R}\left(AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}\right) = \mathcal{R}(AA^{(2)}_{T,S})$.

(iii) This part is clear by $\mathcal{R}(A^{\textcircled{2}}_{T,S}) = \mathcal{R}(A^{\textcircled{2}}_{T,S}A) = T$ and $\mathcal{N}(A^{\textcircled{2}}_{T,S}) = \mathcal{N}\left(AA^{\textcircled{2}}_{T,S}\right) = \mathcal{N}\left(\left(AA^{(2)}_{T,S}\right)^{*}\right)$. $\square$

Remark that, by Lemma 5.1.1, for $m = n$, $k = \mathrm{ind}(A)$ and $A^{(2)}_{T,S} = A^{\mathrm{D}}$, $A^{\textcircled{\dagger}} = A^{\textcircled{2}}_{T,S} = A^{(2)}_{\mathcal{R}(A^{\mathrm{D}}),\mathcal{N}((A^{\mathrm{D}})^*)} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$. In addition, $\mathrm{ind}(A) = 1$ implies $A^{\textcircled{2}}_{T,S} = A^{\textcircled{\#}} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}((A)^*)}$.

The g-core-EP inverse can be alternatively defined by means of a geometrical approach too.

Theorem 5.1.2. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then $X := A^{\textcircled{2}}_{T,S}$ is the unique solution to the constrained matrix equation*

$$\mathcal{R}(X) \subseteq T \quad \text{and} \quad AX = P_{\mathcal{R}(AA^{(2)}_{T,S})}. \tag{5.2}$$

Proof. By Lemma 5.1.1, notice that (5.2) holds for $A^{\textcircled{2}}_{T,S}$.

Let (5.2) be satisfied for different $X, X_1 \in \mathbb{C}^{n\times m}$. Because

$$A(X - X_1) = P_{\mathcal{R}(AA^{(2)}_{T,S})} - P_{\mathcal{R}(AA^{(2)}_{T,S})} = 0,$$

we see $A^{(2)}_{T,S}A(X - X_1) = 0$ and so $\mathcal{R}(X - X_1) \subseteq \mathcal{N}(A^{(2)}_{T,S}A)$. Then, the inclusions $\mathcal{R}(X) \subseteq T = \mathcal{R}(A^{(2)}_{T,S}A)$ and $\mathcal{R}(X_1) \subseteq \mathcal{R}(A^{(2)}_{T,S}A)$ yield $\mathcal{R}(X - X_1) \subseteq \mathcal{N}(A^{(2)}_{T,S}A) \cap \mathcal{R}(A^{(2)}_{T,S}A) = \{0\}$. So, $X = X_1$, i.e. $A^{\textcircled{2}}_{T,S}$ is the unique solution to the constrained system (5.2). $\square$

Using Theorem 5.1.2, we can deduce the following:

(i) if $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$, the core-EP inverse of A is the unique matrix satisfying

$$\mathcal{R}(X) \subseteq \mathcal{R}(A^k) \quad \text{and} \quad AX = P_{\mathcal{R}(A^k)};$$

(ii) [2, Definition 1] if $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$, the unique solution to

$$\mathcal{R}(X) \subseteq \mathcal{R}(A) \quad \text{and} \quad AX = P_{\mathcal{R}(A)},$$

is just the core inverse $A^{\textcircled{\#}}$ of A.

Several characterizations for the g-core-EP inverse are proposed in Theorem 5.1.3.

Theorem 5.1.3. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The following statements are mutually equivalent for $X \in \mathbb{C}^{n\times m}$:*

(i) $X = A^{\textcircled{2}}_{T,S}$;

(ii) $AXA = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A, \quad AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger},$
$XAX = X$ *and* $XA = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A$;

(iii) $AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ *and* $X = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AX$;

(iv) $\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = \left(AA^{(2)}_{T,S}\right)^{\dagger}$ *and* $A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = X$;

(v) $AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ *and* $A^{(2)}_{T,S}AX = X$;

(vi) $XA = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A$ *and* $XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X$;

(vii) $XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X$ *and* $XAA^{(2)}_{T,S} = A^{(2)}_{T,S}$;

(viii) $A^{(2)}_{T,S}AXAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X$ *and* $AXAA^{(2)}_{T,S} = AA^{(2)}_{T,S}$;

(ix) $A^{(2)}_{T,S}AXAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X$ *and* $A^{(2)}_{T,S}AXAA^{(2)}_{T,S} = A^{(2)}_{T,S}$;

(x) $A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*} = XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*}$ *and* $XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X$;

(xi) $XAA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}A = AA^{(2)}_{T,S}A,$
$AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ *and* $XAA^{(2)}_{T,S}A = A^{(2)}_{T,S}A$;

(xii) $XAA^{(2)}_{T,S}AX = X, \quad AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ *and* $XAA^{(2)}_{T,S}A = A^{(2)}_{T,S}A$.

Proof. (i)$\Rightarrow$ (ii): On the basis of $X = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$, one can verify $AXA = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A$. The rest of the proof is evident by Theorem 5.1.1.

(ii) $\Rightarrow$ (iii): It is observable that $A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = XAX = X$.

(iii) $\Rightarrow$ (iv): The assumption $AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ initiates $\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = \left(AA^{(2)}_{T,S}\right)^{\dagger}$.

(iv) $\Rightarrow$ (i): This implication follows by $A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = X$.

(iii) $\Rightarrow$ (v): Using $A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = X$, we obtain

$$A^{(2)}_{T,S}AX = (A^{(2)}_{T,S}AA^{(2)}_{T,S})\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AX = X.$$

(v) $\Rightarrow$ (i): Note that $X = A^{(2)}_{T,S}(AX) = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$.

(ii) $\Rightarrow$ (vi): We have that $X = X(AX) = XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$.

(vi) $\Rightarrow$ (vii): By $XA = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A$ and $A^{(2)}_{T,S} = A^{(2)}_{T,S}AA^{(2)}_{T,S}$, we get $XAA^{(2)}_{T,S} = A^{(2)}_{T,S}$.

(vii) $\Rightarrow$ (i): Since $X = XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ and $XAA^{(2)}_{T,S} = A^{(2)}_{T,S}$, one can conclude $X = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$.

(i) $\Rightarrow$ (viii): The definition $X := A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ yields

$$\begin{aligned} A^{(2)}_{T,S}AXAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} &= A^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} \\ &= A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X \end{aligned}$$

and

$$AXAA^{(2)}_{T,S} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S} = AA^{(2)}_{T,S}.$$

(viii) $\Rightarrow$ (ix): If $AXAA^{(2)}_{T,S} = AA^{(2)}_{T,S}$, then $A^{(2)}_{T,S}AXAA^{(2)}_{T,S} = A^{(2)}_{T,S}AA^{(2)}_{T,S} = A^{(2)}_{T,S}$.

(ix) $\Rightarrow$ (i): Because $X = A^{(2)}_{T,S}AXAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ and $A^{(2)}_{T,S}AXAA^{(2)}_{T,S} = A^{(2)}_{T,S}$, we see that $X = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$.

(vii) $\Rightarrow$ (x): Multiplying $XAA^{(2)}_{T,S} = A^{(2)}_{T,S}$ from the right hand side by $\left(AA^{(2)}_{T,S}\right)^{*}$, notice that $XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*}$.

(x) $\Rightarrow$ (vii): The hypothesis $XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*}$ gives

$$\begin{aligned} XAA^{(2)}_{T,S} &= XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S} = \left(XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*}\right)\left(\left(AA^{(2)}_{T,S}\right)^{\dagger}\right)^{*} \\ &= A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{*}\left(\left(AA^{(2)}_{T,S}\right)^{\dagger}\right)^{*} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S}\right) \\ &= A^{(2)}_{T,S}AA^{(2)}_{T,S} = A^{(2)}_{T,S}. \end{aligned}$$

(ii) $\Rightarrow$ (xi): Using $AX = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ and $XA = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A$, we get

$$AA^{(2)}_{T,S}AX = AA^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$$

and

$$XAA^{(2)}_{T,S}A = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S}A = A^{(2)}_{T,S}A.$$

Now,

$$AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}A = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S}A = AA^{(2)}_{T,S}A$$

and

$$XAA^{(2)}_{T,S}AX = A^{(2)}_{T,S}AX = X,$$

by the equivalence between (i) and (ii).

(xi) $\Rightarrow$ (xii): It is evident.

(xii) $\Rightarrow$ (i): This implication follows by

$$\begin{aligned} X &= (XAA^{(2)}_{T,S}A)X = A^{(2)}_{T,S}AX = A^{(2)}_{T,S}(AA^{(2)}_{T,S}AX) \\ &= A^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}. \end{aligned}$$

The proof is finished. $\square$

Corollary 5.1.2. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then*

(i) $A^{②}_{T,S} \in (AA^{(2)}_{T,S}A)\{1,2,3\}$;

(ii) $A^{②}_{T,S} = \left(AA^{(2)}_{T,S}A\right)^{\dagger}$ *if and only if* $\left(A^{(2)}_{T,S}A\right)^{*} = A^{(2)}_{T,S}A$;

(iii) *for $m = n$,* $A^{②}_{T,S} = \left(AA^{(2)}_{T,S}A\right)^{\#}$ *if and only if* $AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{(2)}_{T,S}A$.

Proof. Follows from Theorem 5.1.3, part (xi). $\square$

The g-core-EP inverse $A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$ can be further characterized under the assumptions $\mathcal{R}(U) = E$ and $\mathcal{N}(V) = F$ for some $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}^{l\times m}$. The notation $\Omega_{U,V} := A^{(2)}_{\mathcal{R}(U),\mathcal{N}(V)}$ will be used in order to simplify presentation.

Theorem 5.1.4. *Let $U \in \mathbb{C}^{n\times k}$, $V \in \mathbb{C}^{l\times m}$ and $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(U),\mathcal{N}(V)}$. Then the next assertions are equivalent:*

(i) *$A^{②}_{T,S}$ just coincides with $X \in \mathbb{C}^{n\times m}$, defined by*

$$X := \Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = U(VAU)^{(1)}V\left(AU(VAU)^{(1)}V\right)^{\dagger};$$

(ii) $VAX = V\left(A\Omega_{U,V}\right)^{\dagger}$ *and* $\Omega_{U,V}AX = X$;

(iii) $VAXA\Omega_{U,V} = V$ *and* $\Omega_{U,V}AXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$;

(iv) $VAXA\Omega_{U,V}(A\Omega_{U,V})^{*} = V(A\Omega_{U,V})^{*}$ *and* $\Omega_{U,V}AXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$;

(v) $XA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$ *and* $XAU = U$;

(vi) $\Omega_{U,V}AXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$ *and* $AXAU = AU$;

(vii) $\Omega_{U,V}AXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$ *and* $A^{*}AXAU = A^{*}AU$.

Proof. The equality $\mathcal{R}(\Omega_{U,V}) = \mathcal{R}(U)$ implies, for $U^{(1)} \in U\{1\}$,

$$\Omega_{U,V} = UU^{(1)}\Omega_{U,V} \quad \text{and} \quad \Omega_{U,V}AU = U.$$

From $\mathcal{N}(\Omega_{U,V}) = \mathcal{N}(V)$, we get, for $V^{(1)} \in V\{1\}$,

$$\Omega_{U,V} = \Omega_{U,V}V^{(1)}V \quad \text{and} \quad VA\Omega_{U,V} = V.$$

(i) $\Rightarrow$ (ii): Applying $X = \Omega_{U,V}(A\Omega_{U,V})^{\dagger}$, we conclude

$$\Omega_{U,V}AX = \Omega_{U,V}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = \Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$$

and

$$VAX = (VA\Omega_{U,V})(A\Omega_{U,V})^{\dagger} = V\left(A\Omega_{U,V}\right)^{\dagger}.$$

(ii) $\Rightarrow$ (i): Notice that $\Omega_{U,V}AX = X$ and $VAX = V\left(A\Omega_{U,V}\right)^{\dagger}$ give

$$\begin{aligned} X &= \Omega_{U,V}AX = \Omega_{U,V}V^{(1)}(VAX) = (\Omega_{U,V}V^{(1)}V)(A\Omega_{U,V})^{\dagger} \\ &= \Omega_{U,V}(A\Omega_{U,V})^{\dagger}. \end{aligned}$$

(i) $\Rightarrow$ (iii): Because $X = \Omega_{U,V}(A\Omega_{U,V})^{\dagger}$, it follows that

$$VAXA\Omega_{U,V} = VA\Omega_{U,V}(A\Omega_{U,V})^{\dagger}A\Omega_{U,V} = VA\Omega_{U,V} = V$$

and

$$\begin{aligned} &\Omega_{U,V}AXA\Omega_{U,V}(A\Omega_{U,V})^{\dagger} \\ &= \Omega_{U,V}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}A\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} \\ &= \Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X. \end{aligned}$$

(iii) $\Rightarrow$ (iv): Obviously.

(iv) $\Rightarrow$ (i): From the assumptions
$VAXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{*} = V(A\Omega_{U,V})^{*}$ and $\Omega_{U,V}AXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X$,
we get

$$\begin{aligned} X &= \Omega_{U,V}AXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} \\ &= \Omega_{U,V}V^{(1)}\left(VAXA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{*}\right)\left((A\Omega_{U,V})^{\dagger}\right)^{*}\left(A\Omega_{U,V}\right)^{\dagger} \\ &= (\Omega_{U,V}V^{(1)}V)(A\Omega_{U,V})^{*}\left((A\Omega_{U,V})^{\dagger}\right)^{*}\left(A\Omega_{U,V}\right)^{\dagger} \\ &= \Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger}. \end{aligned}$$

(i) $\Rightarrow$ (v): By Theorem 5.1.3, it follows

$$XA\Omega_{U,V} = \Omega_{U,V}, \quad XA\Omega_{U,V}\left(A\Omega_{U,V}\right)^{\dagger} = X.$$

Using $\Omega_{U,V}AU = U$, one obtains

$$XAU = (XA\Omega_{U,V})\,AU = \Omega_{U,V}AU = U.$$

(v) $\Rightarrow$ (i): Since $XAU = U$ and $\Omega_{U,V} = UU^{(1)}\Omega_{U,V}$, then

$$\Omega_{U,V} = UU^{(1)}\Omega_{U,V} = XA\left(UU^{(1)}\Omega_{U,V}\right) = XA\Omega_{U,V}.$$

By $XA\Omega_{U,V}(A\Omega_{U,V})^\dagger = X$, we deduce

$$X = (XA\Omega_{U,V})(A\Omega_{U,V})^\dagger = \Omega_{U,V}(A\Omega_{U,V})^\dagger.$$

(i) $\Rightarrow$ (vi): Clearly, it is observable that $X = \Omega_{U,V}(A\Omega_{U,V})^\dagger$ in conjunction with $\Omega_{U,V}AU = U$ imply

$$\begin{aligned}&\Omega_{U,V}AXA\Omega_{U,V}(A\Omega_{U,V})^\dagger\\ &\quad = \Omega_{U,V}A\Omega_{U,V}(A\Omega_{U,V})^\dagger A\Omega_{U,V}(A\Omega_{U,V})^\dagger\\ &\quad = \Omega_{U,V}(A\Omega_{U,V})^\dagger = X.\end{aligned}$$

and

$$\begin{aligned}AXAU &= A\Omega_{U,V}(A\Omega_{U,V})^\dagger AU\\ &= A\Omega_{U,V}(A\Omega_{U,V})^\dagger A\Omega_{U,V}AU = AU.\end{aligned}$$

(vi) $\Rightarrow$ (vii): This implication is clear.

(vii) $\Rightarrow$ (i): If we multiply $A^*AXAU = A^*AU$ from the left hand side by $(A^\dagger)^*$, we get $AXAU = AU$. Multiplying the previous equality by $U^{(1)}\Omega_{U,V}$ from the right hand side, it is possible to derive $AXA\Omega_{U,V} = A\Omega_{U,V}$. So,

$$\Omega_{U,V} = \Omega_{U,V}(A\Omega_{U,V}) = \Omega_{U,V}AXA\Omega_{U,V}.$$

Now, using $\Omega_{U,V}AXA\Omega_{U,V}(A\Omega_{U,V})^\dagger = X$, one obtains

$$\Omega_{U,V}(A\Omega_{U,V})^\dagger = \Omega_{U,V}AXA\Omega_{U,V}(A\Omega_{U,V})^\dagger = X,$$

which finishes the proof. $\square$

Theorem 5.1.5 gives some expressions for the Moore-Penorse inverse of the g-core-EP inverse.

Theorem 5.1.5. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then*

$$(A^{\textcircled{2}}_{T,S})^\dagger = AA^{(2)}_{T,S}(A^{(2)}_{T,S})^\dagger = P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right),S}(A^{(2)}_{T,S})^\dagger = AP_{\mathcal{R}(A^{(2)}_{T,S})}.$$

Proof. We easily verify that $(A^{\textcircled{2}}_{T,S})^\dagger = AA^{(2)}_{T,S}(A^{(2)}_{T,S})^\dagger$, by the definition of the Moore-Penrose inverse and $A^{\textcircled{2}}_{T,S} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^\dagger$. Since $AA^{(2)}_{T,S} = P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right),S}$ and $A^{(2)}_{T,S}(A^{(2)}_{T,S})^\dagger = P_{\mathcal{R}(A^{(2)}_{T,S})}$, the rest is evident. $\square$

As a consequence of Theorem 5.1.5, in Corollary 5.1.3 we obtain representations for g-core-EP inverses.

Corollary 5.1.3. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then*

$$A^{\textcircled{2}}_{T,S} = \left(AA^{(2)}_{T,S}(A^{(2)}_{T,S})^\dagger\right)^\dagger = \left(AP_{\mathcal{R}(A^{(2)}_{T,S})}\right)^\dagger = \left(P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right),S}(A^{(2)}_{T,S})^\dagger\right)^\dagger.$$

For the g-core-EP inverse, we investigate maximal classes of matrices for which its representation is satisfied.

Theorem 5.1.6. [113] *Let $\Lambda \in \mathbb{C}^{n\times m}$ and $A \in \mathbb{C}^{m\times n}_{T,S}$. The following propositions are equivalent:*

(i) $A^{\textcircled{2}}_{T,S} = \Lambda(A\Lambda)^\dagger$;

(ii) $A\Lambda(A\Lambda)^\dagger = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^\dagger$ *and* $(I - A^{(2)}_{T,S}A)\Lambda(A\Lambda)^\dagger = 0$;

(iii) $\mathcal{R}(A\Lambda) = \mathcal{R}\left(AA^{(2)}_{T,S}\right)$ *and* $\mathcal{R}(\Lambda(A\Lambda)^*) \subseteq T$.

Proof. (i) $\Rightarrow$ (ii): Because $A^{\textcircled{2}}_{T,S} = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^\dagger = \Lambda(A\Lambda)^\dagger$, we firstly get

$$AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^\dagger = A\Lambda(A\Lambda)^\dagger.$$

Further,

$$(I - A^{(2)}_{T,S}A)\Lambda(A\Lambda)^\dagger = (I - A^{(2)}_{T,S}A)A^{\textcircled{2}}_{T,S} = (I - A^{(2)}_{T,S}A)A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^\dagger = 0.$$

(ii) $\Rightarrow$ (iii): From $A\Lambda(A\Lambda)^{\dagger} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}$, it follows that

$$\mathcal{R}(A\Lambda) = \mathcal{R}(A\Lambda(A\Lambda)^{\dagger}) = \mathcal{R}(AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}) = \mathcal{R}\left(AA^{(2)}_{T,S}\right).$$

Also, by $(I - A^{(2)}_{T,S}A)\Lambda(A\Lambda)^{\dagger} = 0$, we observe that

$$\mathcal{R}(\Lambda(A\Lambda)^{*}) = \mathcal{R}(\Lambda(A\Lambda)^{\dagger}) \subseteq \mathcal{N}(I - A^{(2)}_{T,S}A) = T.$$

(iii) $\Rightarrow$ (i): Notice that

$$\mathcal{R}(A\Lambda(A\Lambda)^{\dagger}) = \mathcal{R}(A\Lambda) = \mathcal{R}\left(AA^{(2)}_{T,S}\right) = \mathcal{R}\left(AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}\right)$$

implies

$$AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A\Lambda(A\Lambda)^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}.$$

and

$$A\Lambda(A\Lambda)^{\dagger} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A\Lambda(A\Lambda)^{\dagger}.$$

Therefore,

$$\begin{aligned} A\Lambda(A\Lambda)^{\dagger} &= AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}A\Lambda(A\Lambda)^{\dagger} = \left(AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}\right)^{*} \\ &= AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}. \end{aligned}$$

We observe that

$$\mathcal{R}(\Lambda(A\Lambda)^{\dagger}) = \mathcal{R}(\Lambda(A\Lambda)^{*}) \subseteq T = \mathcal{N}(I - A^{(2)}_{T,S}A)$$

gives $(I - A^{(2)}_{T,S}A)\Lambda(A\Lambda)^{\dagger} = 0$. Thus,

$$\Lambda(A\Lambda)^{\dagger} = A^{(2)}_{T,S}A\Lambda(A\Lambda)^{\dagger} = A^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{②}_{T,S}.$$

The proof is complete. $\square$

5.1.2 Characterizations of *g-core-EP inverse

The *g-core-EP inverse is defined as the dual case of the g-core-EP inverse.

Theorem 5.1.7. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the matrix equations*

$$XAX = X, \qquad AX = A\left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S} \qquad \text{and} \qquad XA = \left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S}A$$

have the unique solution $X := \left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S}$.

Definition 5.1.2. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The* generalized core-EP (or *g-core-EP) inverse *of A is defined as*

$$A^{T,S}_{②} := \left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S}.$$

Some particular cases of the *g-core-EP inverse are listed:

(i) When $m = n$, $k = \mathrm{ind}(A)$ and $A^{(2)}_{T,S} = A^{\mathrm{D}}$, the *g-core-EP inverse coincides with the *core-EP inverse:

$$A^{T,S}_{②} = (A^{\mathrm{D}}A)^{\dagger}A^{\mathrm{D}} = P_{\mathcal{R}((AA^{\mathrm{D}})^{*})}A^{\mathrm{D}} = P_{\mathcal{R}((A^{k})^{*})}A^{\mathrm{D}} = (A^{k})^{\dagger}A^{k}A^{\mathrm{D}} = A_{\textcircled{\dagger}}.$$

(ii) If $k = 1$ in the part (i), then $A^{T,S}_{(2)} = A^{\#}$ and the generalized *core-EP inverse becomes the dual core (or *core) inverse.

Corollary 5.1.4. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then*

$$A^{T,S}_{②} = P_{\mathcal{R}\left(\left(A^{(2)}_{T,S}A\right)^{*}\right)}A^{(2)}_{T,S}.$$

We also investigate projections involving *g-core-EP inverse.

Lemma 5.1.2. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then:*

(i) $A_{\circledtwo}^{T,S}A$ *is the orthogonal projector onto* $\mathcal{R}\left(\left(A_{T,S}^{(2)}A\right)^*\right)$*;*

(ii) $AA_{\circledtwo}^{T,S}$ *is a projector onto* $\mathcal{R}(A\left(A_{T,S}^{(2)}A\right)^*)$ *along* S*;*

(iii) $A_{\circledtwo}^{T,S}=A^{(2)}_{\mathcal{R}(\left(A_{T,S}^{(2)}A\right)^*),S}$.

By from a geometrical point of view, the *g-core-EP inverse is introduced now.

Theorem 5.1.8. [113] *Let* $A\in\mathbb{C}_{T,S}^{m\times n}$*. Then* $X=A_{\circledtwo}^{T,S}$ *is the unique solution to the following constrained matrix equation:*

$$\mathcal{R}(X)\subseteq\mathcal{R}\left(\left(A_{T,S}^{(2)}A\right)^*\right)\quad \textit{and}\quad AX=P_{\mathcal{R}(A\left(A_{T,S}^{(2)}A\right)^*),S}.$$

We present a few characterizations of the *g-core-EP inverse too.

Theorem 5.1.9. [113] *The following assertions are equivalent for* $A\in\mathbb{C}_{T,S}^{m\times n}$ *and* $X\in\mathbb{C}^{n\times m}$*:*

(i) X *is the *g-core-EP inverse of* A*;*

(ii) $AXA=A\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A,\qquad XAX=X,$
$AX=A\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}\qquad$ *and* $\qquad XA=\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A;$

(iii) $XA\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}=X$ *and* $XA=\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A;$

(iv) $XA\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}=X$ *and* $\left(A_{T,S}^{(2)}A\right)^\dagger=XA\left(A_{T,S}^{(2)}A\right)^\dagger;$

(v) $XA=\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A$ *and* $XAA_{T,S}^{(2)}=X;$

(vi) $\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}AX=X$ *and* $AX=A\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)};$

(vii) $\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}AX=X$ *and* $A_{T,S}^{(2)}AX=A_{T,S}^{(2)};$

(viii) $\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}AXAA_{T,S}^{(2)}=X$ *and* $A_{T,S}^{(2)}AXA=A_{T,S}^{(2)}A;$

(ix) $\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}AXAA_{T,S}^{(2)}=X$ *and* $A_{T,S}^{(2)}AXAA_{T,S}^{(2)}=A_{T,S}^{(2)};$

(x) $\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}AX=X$ *and* $\left(A_{T,S}^{(2)}A\right)^*A_{T,S}^{(2)}=\left(A_{T,S}^{(2)}A\right)^*A_{T,S}^{(2)}AX;$

(xi) $XAA_{T,S}^{(2)}AX=X,\quad AA_{T,S}^{(2)}AXAA_{T,S}^{(2)}A=AA_{T,S}^{(2)}A,$
$AA_{T,S}^{(2)}AX=AA_{T,S}^{(2)}$ *and* $XAA_{T,S}^{(2)}A=\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A;$

(xii) $XAA_{T,S}^{(2)}AX=X,\ AA_{T,S}^{(2)}AX=AA_{T,S}^{(2)}$ *and* $XAA_{T,S}^{(2)}A=\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A.$

Several additional properties of the *g-core-EP inverse are given in Corollary 5.1.5.

Corollary 5.1.5. [113] *Let* $A\in\mathbb{C}_{T,S}^{m\times n}$*. Then*

(i) $A_{\circledtwo}^{T,S}\in(AA_{T,S}^{(2)}A)\{1,2,4\};$

(ii) $A_{\circledtwo}^{T,S}=(AA_{T,S}^{(2)}A)^\dagger$ *if and only if* $\left(AA_{T,S}^{(2)}\right)^*=AA_{T,S}^{(2)};$

(iii) *for* $m=n$*,* $A_{\circledtwo}^{T,S}=(AA_{T,S}^{(2)}A)^{\#}$ *if and only if* $AA_{T,S}^{(2)}=\left(A_{T,S}^{(2)}A\right)^\dagger A_{T,S}^{(2)}A.$

We also consider the *g-core-EP inverse of the form $(\Omega_{U,V}A)^\dagger\Omega_{U,V}$.

Theorem 5.1.10. *Let* $U\in\mathbb{C}^{n\times k}$*,* $V\in\mathbb{C}^{l\times m}$ *and* $A\in\mathbb{C}^{m\times n}_{\mathcal{R}(U),\mathcal{N}(V)}$*. Then the next statements are equivalent:*

(i) *the *g-core-EP inverse of* A *is* $X\in\mathbb{C}^{n\times m}$*, defined by*

$$X:=(\Omega_{U,V}A)^\dagger\,\Omega_{U,V}=\left(U(VAU)^{(1)}VA\right)^\dagger U(VAU)^{(1)}V.$$

(ii) $XAU=(\Omega_{U,V}A)^\dagger\,U$ *and* $XA\Omega_{U,V}=X;$

(iii) $\Omega_{U,V}AXAU=U$ *and* $(\Omega_{U,V}A)^\dagger\,\Omega_{U,V}AXA\Omega_{U,V}=X;$

(iv) $(\Omega_{U,V}A)^*\,\Omega_{U,V}AXAU=(\Omega_{U,V}A)^*\,U$ *and* $(\Omega_{U,V}A)^\dagger\,\Omega_{U,V}AXA\Omega_{U,V}=X$;

(v) $(\Omega_{U,V}A)^\dagger\,\Omega_{U,V}AX=X$ *and* $VAX=V$;

(vi) $(\Omega_{U,V}A)^\dagger\,\Omega_{U,V}AXA\Omega_{U,V}=X$ *and* $VAXA=VA$;

(vii) $(\Omega_{U,V}A)^\dagger\,\Omega_{U,V}AXA\Omega_{U,V}=X$ *and* $VAXAA^*=VAA^*$.

We can establish formulae for the Moore-Penorse inverse of the generalized *core–EP inverse.

Theorem 5.1.11. *Let* $A\in\mathbb{C}^{m\times n}_{T,S}$. *Then*

$$(A^{T,S}_{②})^\dagger=(A^{(2)}_{T,S})^\dagger A^{(2)}_{T,S}A=P_{\mathcal{R}\left((A^{(2)}_{T,S})^*\right)}A=(A^{(2)}_{T,S})^\dagger P_{T,\mathcal{N}\left(A^{(2)}_{T,S}A\right)}.$$

Corollary 5.1.6. [113] *Let* $A\in\mathbb{C}^{m\times n}_{T,S}$. *Then*

$$A^{T,S}_{②}=\left((A^{(2)}_{T,S})^\dagger A^{(2)}_{T,S}A\right)^\dagger=\left((A^{(2)}_{T,S})^\dagger P_{T,\mathcal{N}\left(A^{(2)}_{T,S}A\right)}\right)^\dagger=\left(P_{\mathcal{R}\left((A^{(2)}_{T,S})^*\right)}A\right)^\dagger.$$

We give maximal classes for representing the *g-core-EP inverse in the most general form.

Theorem 5.1.12. [113] *Let* $\Lambda\in\mathbb{C}^{n\times m}$ *and let* $A\in\mathbb{C}^{m\times n}_{T,S}$. *The next conditions are equivalent:*

(i) $A^{T,S}_{②}=(\Lambda A)^\dagger\Lambda$;

(ii) $\left(A^{(2)}_{T,S}A\right)^\dagger A^{(2)}_{T,S}A=(\Lambda A)^\dagger\Lambda A$ *and* $(\Lambda A)^\dagger\Lambda(I-AA^{(2)}_{T,S})=O$;

(iii) $\mathcal{R}((AV)^*)=\mathcal{R}\left(\left(A^{(2)}_{T,S}A\right)^*\right)$ *and* $S\subseteq\mathcal{N}((\Lambda A)^*\Lambda)$.

5.2 Integral and limit representations of g-core-EP and *g-core-EP inverses

5.2.1 Integral and limit representations of g-core-EP inverse

In the next theorem, we establish some integral representations for the g-core-EP inverse.

Theorem 5.2.1. [113] *Let* $A\in\mathbb{C}^{m\times n}_{T,S}$.

(i) *If* $G\in\mathbb{C}^{n\times m}$ *with* $\mathcal{N}(G)=S$ *and* $\mathcal{R}(G)=T$, *then*

$$A^{②}_{T,S}=\int_0^\infty \exp\left[-G(GAG)^*GAt\right]G(GAG)^*GP_{\mathcal{R}\left(AA^{(2)}_{T,S}\right)}\,\mathrm{d}t.$$

(ii) *If* $G_1\in\mathbb{C}^{n\times m}$ *with* $\mathcal{N}(G_1)=\mathcal{N}(\left(A^{(2)}_{T,S}A\right)^*)$ *and* $\mathcal{R}(G_1)=T$, *then*

$$A^{②}_{T,S}=\int_0^\infty \exp\left[-G_1(G_1AG_1)^*G_1At\right]G_1(G_1AG_1)^*G_1\,\mathrm{d}t.$$

Proof. (i) According to [165, Theorem 2.2], note that

$$A^{(2)}_{T,S}=\int_0^\infty \exp\left[-G(GAG)^*GAt\right]G(GAG)^*G\,\mathrm{d}t.$$

By Corollary 5.1.1, recall that $A^{②}_{T,S}=A^{(2)}_{T,S}P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right)}$ and the rest is clear.

(ii) From Lemma 5.1.1(iii), we conclude that $A^{②}_{T,S}=A^{(2)}_{T,\mathcal{N}\left(\left(A^{(2)}_{T,S}A\right)^*\right)}$. Using [165, Theorem 2.2], we conclude that (ii) is satisfied. □

Now, we derive the limit representations for the g-core-EP inverse.

Theorem 5.2.2. [113] *Let* $A\in\mathbb{C}^{m\times n}_{T,S}$.

(i) *If* $U\in\mathbb{C}^{n\times s}_s$ *and* $V\in\mathbb{C}^{s\times m}_s$ *with* $\mathcal{N}(V)=S$ *and* $\mathcal{R}(U)=T$, *then*

$$A^{②}_{T,S}=\lim_{t\to 0}U(tI+VAU)^{-1}VP_{\mathcal{R}\left(AA^{(2)}_{T,S}\right)}.$$

(ii) *If* $U_1 \in \mathbb{C}_s^{n\times s}$ *and* $V_1 \in \mathbb{C}_s^{s\times m}$ *with* $\mathcal{N}(V_1) = \mathcal{N}\left(\left(A_{T,S}^{(2)}A\right)^*\right)$ *and* $\mathcal{R}(U_1) = T$*, then*

$$\begin{aligned} A_{T,S}^{②} &= \lim_{t\to 0} U_1(tI + V_1AU_1)^{-1}V_1 \\ &= \lim_{t\to 0}(tI + U_1V_1A)^{-1}U_1V_1 = \lim_{t\to 0} U_1V_1(tI + AU_1V_1)^{-1}. \end{aligned}$$

Proof. It follows from

$$A_{T,S}^{(2)} = \lim_{t\to 0} U(tI + VAU)^{-1}V,$$

which is known result [76, Theorem 7]. □

5.2.2 Integral and limit representations of *g-core-EP inverse

The integral representations for the *g-core-EP inverse are given in Theorem 5.2.3.

Theorem 5.2.3. [113] *Let* $A \in \mathbb{C}_{T,S}^{m\times n}$.

(i) *If* $G \in \mathbb{C}^{n\times m}$ *with* $\mathcal{N}(G) = S$ *and* $\mathcal{R}(G) = T$*, then*

$$A_{②}^{T,S} = \int_0^\infty P_{\mathcal{R}\left(\left(A_{T,S}^{(2)}A\right)^*\right)} \exp\left[-G(GAG)^*GAt\right]G(GAG)^*G\,\mathrm{d}t.$$

(ii) *If* $G_1 \in \mathbb{C}^{n\times m}$ *with* $\mathcal{N}(G_1) = S$ *and* $\mathcal{R}(G_1) = \mathcal{R}(\left(A_{T,S}^{(2)}A\right)^*)$*, then*

$$A_{②}^{T,S} = \int_0^\infty \exp\left[-G_1(G_1AG_1)^*G_1At\right]G_1(G_1AG_1)^*G_1\,\mathrm{d}t.$$

Some limit representations for the generalized *core–EP inverse are presented in Theorem 5.2.4.

Theorem 5.2.4. [113] *Let* $A \in \mathbb{C}_{T,S}^{m\times n}$.

(i) *If* $U \in \mathbb{C}_s^{n\times s}$ *and* $V \in \mathbb{C}_s^{s\times m}$ *with* $\mathcal{N}(V) = S$ *and* $\mathcal{R}(U) = T$*, then*

$$A_{②}^{T,S} = \lim_{t\to 0} P_{\mathcal{R}\left(\left(A_{T,S}^{(2)}A\right)^*\right)} U(tI + VAU)^{-1}V.$$

(ii) *If* $U_1 \in \mathbb{C}_s^{n\times s}$ *and* $V_1 \in \mathbb{C}_s^{s\times m}$ *with* $\mathcal{N}(V_1) = S$ *and* $\mathcal{R}(U_1) = \mathcal{R}\left(\left(A_{T,S}^{(2)}A\right)^*\right)$*, then*

$$\begin{aligned} A_{②}^{T,S} &= \lim_{t\to 0} U_1(tI + V_1AU_1)^{-1}V_1 \\ &= \lim_{t\to 0}(tI + U_1V_1A)^{-1}U_1V_1 = \lim_{t\to 0} U_1V_1(tI + AU_1V_1)^{-1}. \end{aligned}$$

5.3 Applications of g-core-EP inverses

Applications of g-core-EP and *g-core-EP inverses

The g-core-EP and *g-core-EP inverse can be applied in solving certain systems of linear equations. Theorem 5.3.1 investigates applicability of the g-core-EP inverse.

Theorem 5.3.1. [113] *If* $A \in \mathbb{C}_{T,S}^{m\times n}$*, then the general solution to*

$$A_{T,S}^{(2)}Ax = A_{T,S}^{②}b \tag{5.3}$$

is given by

$$x = A_{T,S}^{②}b + \left(I - A_{T,S}^{(2)}A\right)y, \tag{5.4}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Notice that x of the form (5.4) is a solution to (5.3):

$$A_{T,S}^{(2)}Ax = A_{T,S}^{(2)}AA_{T,S}^{②}b = A_{T,S}^{②}b.$$

Let x be a solution to (5.3). Then, by $A_{T,S}^{②}b = A_{T,S}^{(2)}Ax$, we deduce that x has the form (5.4):

$$x = A_{T,S}^{②}b + x - A_{T,S}^{(2)}Ax = A_{T,S}^{②}b + \left(I - A_{T,S}^{(2)}A\right)x.$$

□

Applying Theorem 5.3.1, we obtain the next results.

Corollary 5.3.1. *If $A \in \mathbb{C}^{n\times n}$ satisfies* $\operatorname{ind}(A) = k$ *then the general solution to*

$$A^{\mathrm{D}}Ax = A^{\textcircled{\dagger}}b$$

is represented by

$$x = A^{\textcircled{\dagger}}b + (I - A^{\mathrm{D}}A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

Corollary 5.3.2. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = 1$. *Then the general solution to*

$$A^{\#}Ax = A^{\textcircled{\#}}b$$

is

$$x = A^{\textcircled{\#}}b + (I - A^{\#}A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

Theorem 5.3.2. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the general solution to*

$$A^{(2)}_{T,S}Ax = A^{(2)}_{T,S}b, \quad b \in \mathcal{R}(AA^{(2)}_{T,S}), \tag{5.5}$$

is given by

$$\begin{aligned} x &= A^{\textcircled{2}}_{T,S}b + \left(I - A^{(2)}_{T,S}A\right)y, \\ &= A^{(2)}_{T,S}b + \left(I - A^{(2)}_{T,S}A\right)y \end{aligned} \tag{5.6}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. If x is represented by (5.6), then

$$A^{(2)}_{T,S}Ax = A^{(2)}_{T,S}AA^{\textcircled{2}}_{T,S}b = A^{(2)}_{T,S}P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right)}b = A^{(2)}_{T,S}b.$$

Hence, x is a solution to (5.5).

On the other hand, assume that x is a solution to (5.5). Using

$$A^{\textcircled{2}}_{T,S}b = A^{(2)}_{T,S}P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right)}b = A^{(2)}_{T,S}b = A^{(2)}_{T,S}Ax,$$

one can conclude that

$$x = A^{\textcircled{2}}_{T,S}b + x - A^{(2)}_{T,S}Ax = A^{\textcircled{2}}_{T,S}b + \left(I - A^{(2)}_{T,S}A\right)x.$$

Thus, the solution x to (5.5) possesses the form (5.6). Since $b \in \mathcal{R}(AA^{(2)}_{T,S})$, we observe the identities $A^{\textcircled{2}}_{T,S}b = A^{(2)}_{T,S}P_{\mathcal{R}\left(AA^{(2)}_{T,S}\right)}b = A^{(2)}_{T,S}b$, which confirm the second identity in (5.6). $\square$

Certain system of linear equations can be solved using the *g-core-EP inverse.

Theorem 5.3.3. [113] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the general solution to*

$$A^{(2)}_{T,S}Ax = A^{(2)}_{T,S}b \tag{5.7}$$

is represented as

$$x = A^{T,S}_{\textcircled{2}}b + \left(I - A^{T,S}_{\textcircled{2}}A\right)y, \tag{5.8}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Since $A^{(2)}_{T,S}AA^{T,S}_{\textcircled{2}} = A^{(2)}_{T,S}$, we see that x given by (5.8), is a solution to (5.7):

$$A^{(2)}_{T,S}Ax = A^{(2)}_{T,S}AA^{T,S}_{\textcircled{2}}b = A^{(2)}_{T,S}b.$$

If x is a solution to (5.7), then

$$A^{T,S}_{\textcircled{2}}b = \left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S}b = \left(A^{(2)}_{T,S}A\right)^{\dagger}A^{(2)}_{T,S}Ax = A^{T,S}_{\textcircled{2}}Ax$$

implies

$$x = A^{T,S}_{\textcircled{2}}b + x - A^{T,S}_{\textcircled{2}}Ax = A^{T,S}_{\textcircled{2}}b + \left(I - A^{T,S}_{\textcircled{2}}A\right)x.$$

So, x has the form (5.8). $\square$

Remark 5.3.1. *Notice that the equation (5.5) is a special case of the equation (5.7). Under the assumption $b \in \mathcal{R}(AA^{(2)}_{T,S})$, we can verify that the general solution (5.8) to (5.7) reduces to the general solution (5.6) to (5.5). Indeed, since the hypothesis $b \in \mathcal{R}(AA^{(2)}_{T,S})$ gives $b = AA^{(2)}_{T,S}u = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} b$, for some $u \in \mathbb{C}^m$, we obtain*

$$\begin{aligned}
A^{T,S}_{\textcircled{2}}b + \left(I - A^{T,S}_{\textcircled{2}}A\right)y &= A^{(2)}_{T,S}b - A^{(2)}_{T,S}b + \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}AA^{(2)}_{T,S}u \\
&\quad + \left(I - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}A\right)y \\
&= A^{(2)}_{T,S}b + \left(I - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}A\right)(y - A^{(2)}_{T,S}u)
\end{aligned}$$

for arbitrary $y \in \mathbb{C}^n$. Choosing $y = A^{(2)}_{T,S}u + (I - A^{(2)}_{T,S}A)z$, $z \in \mathbb{C}^n$, we obtain

$$\begin{aligned}
A^{T,S}_{\textcircled{2}}b + \left(I - A^{T,S}_{\textcircled{2}}A\right)y &= A^{(2)}_{T,S}b + \left(I - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}A\right)(I - A^{(2)}_{T,S}A)z \\
&= A^{(2)}_{T,S}b + (I - A^{(2)}_{T,S}A)z.
\end{aligned}$$

Thus,

$$\left\{A^{T,S}_{\textcircled{2}}b + \left(I - A^{T,S}_{\textcircled{2}}A\right)y : \ y \in \mathbb{C}^n\right\} \subseteq \{A^{(2)}_{T,S}b + (I - A^{(2)}_{T,S}A)z : \ z \in \mathbb{C}^n\}.$$

To prove the converse inclusion, we observe that

$$\begin{aligned}
A^{(2)}_{T,S}b + (I - A^{(2)}_{T,S}A)z &= A^{T,S}_{\textcircled{2}}b - A^{T,S}_{\textcircled{2}}b + A^{(2)}_{T,S}b + (I - A^{(2)}_{T,S}A)z \\
&= A^{T,S}_{\textcircled{2}}b - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}b + A^{(2)}_{T,S}A\left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}b \\
&\quad + (I - A^{(2)}_{T,S}A)z \\
&= A^{T,S}_{\textcircled{2}}b + (I - A^{(2)}_{T,S}A)\left(z - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}b\right)
\end{aligned}$$

for arbitrary $z \in \mathbb{C}^n$. After the replacement

$$z = \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}b + \left(I - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}A\right)y, \quad y \in \mathbb{C}^n,$$

it can be obtained

$$\begin{aligned}
A^{(2)}_{T,S}b + (I - A^{(2)}_{T,S}A)z &= A^{T,S}_{\textcircled{2}}b + (I - A^{(2)}_{T,S}A)\left(I - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}A\right)y \\
&= A^{T,S}_{\textcircled{2}}b + \left(I - \left(A^{(2)}_{T,S}A\right)^{\dagger} A^{(2)}_{T,S}A\right)y.
\end{aligned}$$

Therefore,

$$\left\{A^{T,S}_{\textcircled{2}}b + \left(I - A^{T,S}_{\textcircled{2}}A\right)y : \ y \in \mathbb{C}^n\right\} = \left\{A^{(2)}_{T,S}b + (I - A^{(2)}_{T,S}A)z : \ z \in \mathbb{C}^n\right\}.$$

5.4 Summary

The first aim of this chapter is to solve two constrained matrix approximation problem in Euclidean norm and present their Cramer's rules based on [111]. Based on the core-EP inverse, we establish the unique solution of problem (4.2). Further, applying one well-known expression and one new representation of the core-EP inverse, we obtain two kinds of Cramer's rules to find unique solution to (4.2). Using the W-weighted core-EP inverse, we investigate the constrained matrix approximation problem (4.3). Also, we get two various Cramer's rules for finding the solution to (4.3) by means of the one recent proposed formula and one new expression of W-weighted core-EP inverse. In this way, we solve problems which reduce to the problem proposed in [159] for complex matrices of index one to complex matrices of arbitrary index. Also, we did not used the assumptions which appeared in [12, 155].

Numerical comparison of the proposed usage of the core-EP inverse with classical methods for solving least squares (LS) problems with linear equality constraints show the effectiveness of the proposed strategy based on the usage of the core-EP inverse in solving specific constrained least squares problems. Moreover, two applications in solving linear systems and systems of linear matrix equations are considered. One application investigates particular cases of the considered constrained optimization problem, while the second is application in solving constrained matrix equations. Application in solving constrained matrix equations generalizes already investigated application of the Drazin inverse in solving constrained matrix equations.

Some of possibilities for further work can be generalizations to more general structures, such as Hilbert spaces or Banach algebras.

Since composite outer inverses do not involve the core-EP inverse, it seems reasonable to introduce and study extension of the core-EP inverse for rectangular matrices, using proper compositions of outer inverses and the Moore-Penorse inverse. Thus, we define the g-core-EP inverse for rectangular matrices as a wider class of outer inverses including the core-EP inverse. Several useful characterizations and various representations of the g-core-EP inverse are given as well as its integral and limit representations. The Moore-Penrose of the g-core-EP inverse is considered. The *g-core-EP inverse for an arbitrary matrix is investigated too as generalization of *core-EP inverse for a square matrix. It is verified that some systems of equations can be solved using the g-core-EP and *g-core-EP inverses.

We really believe that research about the g-core-EP and *g-core-EP inverses, which can be found in [113], will be very popular in the next years. First, we believe that the generalizations introduced will initiate future more general research. In addition, we believe that the new properties and characterizations of the introduced inverses will be an interesting topic for future research. Several perspectives for further examinations can be pronounced as follows.

1. It will be challenging problem to consider perturbation results on g-core-EP and *g-core-EP inverses for further research.

2. Iterative methods for computing g-core-EP and *g-core-EP inverses can be interesting research area.

3. An extension of g-core-EP and *g-core-EP inverses from the matrix case to Hilbert spaces operators can be considered in future research.

4. The g-core-EP and *g-core-EP inverses of tensors could be studied too.

5.5 Generalizations of OMP, OMP and MPOMP inverses

In this section we investigate various extensions of the OMP inverses defined by the expression $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$, MPO inverses $A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ and MPOMP inverses $A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$, where $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$. The motivation arises from the famous Urquhart representation which was proposed in [150] and confirmed in [156, Theorem 1.3.3] and [8, Theorem 13, P. 72]. We will use the following part of this statement which is useful in representations of generalized inverses with predefined range and/or null space. According to standard notation, $(CAB)^{(1)}$ is a fixed but arbitrary element of $(CAB)\{1\}$. The relationship $\mathrm{rank}(A_1) = \cdots = \mathrm{rank}(A_k)$ between the matrices $A_1,\dots,A_k$ of appropriate order will be denoted by $\mho_{A_1,\dots,A_k}$, while $\mho^B_{A_1,\dots,A_k}$ stands for $\mathrm{rank}(A_1) = \cdots = \mathrm{rank}(A_k) < \mathrm{rank}(B)$. The set of matrices $A_1,\dots,A_k$ satisfying $\mho_{A_1,\dots,A_k}$ or $\mho^B_{A_1,\dots,A_k}$ will be denoted by $\Theta_{A_1,\dots,A_k}$ and $\Theta^B_{A_1,\dots,A_k}$, respectively.

Proposition 5.5.1. (Urquhart formula).
For arbitrary $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$, it follows

$$\Phi_1 := B(CAB)^{(1)}C \begin{cases} \in A\{2\}_{\mathcal{R}(B),*} \Longleftrightarrow \mho_{CAB,B} \\ \in A\{2\}_{*,\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,C} \\ = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,B,C} \\ = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,B,C,A}, \end{cases} \tag{5.9}$$

where $(CAB)^{(1)}$ is an arbitrary but fixed inner inverse of CAB.

It is observable that expressions $\Phi_1 := B(CAB)^{(1)}C$ are outer inverses $A^{(2)}_{\mathcal{R}(B),*}$ with known only range in the case $\mho_{CAB,B}$, outer inverses $A^{(2)}_{*,\mathcal{N}(C)}$ with known only kernel in the case $\mho_{CAB,C}$ and becomes outer inverses with defined both range and null space $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ in cases $\mho_{CAB,B,C}$ and $\mho_{CAB,B,C,A}$. Now, our extension is clear and based on the replacement of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ by the more general expression Φ_1 inside the expressions included in definitions of OMP, MPO and MPOMP inverses. Obtained inverses will be termed as Φ_1-OMP, Φ_1-MPO and Φ_1-MPOMP inverses and marked by the common term Φ_1*-composite outer inverses*. Our general intention is to show that these generalized inverses belong to classes $A^{(2)}_{\mathcal{R}(U),*}$ and/or $A^{(2)}_{*,\mathcal{N}(V)}$, where U and V are appropriate matrices. Generalized inverses based on the replacements of

$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ in composite outer inverses by the expressions $\Phi_2 := B(CAB)^{(2)}C \in A\{2\}$ will be termed as Φ_2-*composite outer inverses*, and divided into Φ_2-OMP, Φ_2-MPO and Φ_2-MPOMP inverses. These results are presented in [137].

The Urquhart representation of generalized inverses motivated us to consider generalizations of OMP, MPO and MPOMP inverses which are based on the replacement of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ by the more general expression Φ_1. In this way, composite outer inverses will be defined using outer inverses with prescribed range $A\{2\}_{\mathcal{R}(U),*}$ and/or null space $A\{2\}_{*,\mathcal{N}(V)}$. Our goal will be investigation of matrices U and V. The set of $m \times n$ matrices with rational expressions over $\mathbb{C}$ with unknowns $\mathbf{x} = \{x_1, \dots, x_k\}$ will be denoted by $\mathbb{C}(\mathbf{x})^{m\times n}$. The environment $A \in \mathbb{C}(\mathbf{x})^{m\times n}$, $B \in \mathbb{C}(\mathbf{x})^{n\times p}$, $C \in \mathbb{C}(\mathbf{x})^{q\times m}$ will be assumed in the rest of the manuscript. The situation when A, B, C are fixed will be marked by $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$. If $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ is fixed and B, C are changeable, the notation $\{B, C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}$ will be used.

For a given $A \in \mathbb{C}(\mathbf{x})^{m\times n}$, the sets of OMP, MPO and MPOMP inverses are defined, respectively, by

$$\begin{aligned}
A_{\Omega_\dagger} &= \left\{A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A_{\dagger\Omega} &= \left\{A^\dagger A A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A_{\dagger\Omega_\dagger} &= \left\{A^\dagger A A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\}.
\end{aligned}$$

Definition 5.5.1. **(a)** *The sets of Φ_1-composite outer inverses of $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ include Φ_1-OMP, Φ_1-MPO and Φ_1-MPOMP inverses which are defined, respectively, by*

$$\begin{aligned}
A_{\Gamma_\dagger} &= \left\{\Phi_1 AA^\dagger |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A_{\dagger\Gamma} &= \left\{A^\dagger A\Phi_1 |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A_{\dagger\Gamma_\dagger} &= \left\{A^\dagger A\Phi_1 AA^\dagger |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\},
\end{aligned}$$

where $\Phi_1 := B(CAB)^{(1)}C$ and $(CAB)^{(1)}$ is a fixed but arbitrary element of $(CAB)\{1\}$.

(b) *The sets of Φ_2-composite outer inverses of $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ include Φ_2-OMP, Φ_2-MPO and Φ_2-MPOMP inverses which are defined, respectively, by*

$$\begin{aligned}
A_{\Gamma_\dagger\{2\}} &= \left\{\Phi_2 AA^\dagger |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A_{\dagger\Gamma\{2\}} &= \left\{A^\dagger A\Phi_2 |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A_{\dagger\Gamma_\dagger\{2\}} &= \left\{A^\dagger A\Phi_2 AA^\dagger |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\},
\end{aligned}$$

where $\Phi_2 := B(CAB)^{(2)}C$ and $(CAB)^{(2)}$ is a fixed but arbitrary element of $(CAB)\{2\}$.

The following sets will be useful in further presentation.

Definition 5.5.2. *Outer inverses of $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ with prescribed range and/or kernel are defined by the following sets:*

$$\begin{aligned}
A\{2:\mathcal{R}(B),*\} &= \left\{\Phi_1 |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,B}\right\};\\
A\{2:*,\mathcal{N}(C)\} &= \left\{\Phi_1 |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,C}\right\};\\
A\{2:\mathcal{R}(B),\mathcal{N}(C)\} &= \left\{\Phi_1 |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,B,C}\right\};\\
A\{1,2:\mathcal{R}(B),\mathcal{N}(C)\} &= \left\{\Phi_1 |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,B,C,A}\right\}.
\end{aligned}$$

Notice that the proof of Lemma 5.5.1 is based on Proposition 5.5.1.

Lemma 5.5.1. *The sets of outer inverses of $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ with prescribed range and/or kernel are equal to the following sets:*

$$\begin{aligned}
A\{2:\mathcal{R}(B),*\} &= \left\{A^{(2)}_{\mathcal{R}(B),*} |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A\{2:*,\mathcal{N}(C)\} &= \left\{A^{(2)}_{*,\mathcal{N}(C)} |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A\{2:\mathcal{R}(B),\mathcal{N}(C)\} &= \left\{A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\};\\
A\{1,2:\mathcal{R}(B),\mathcal{N}(C)\} &= \left\{A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} |\ \{B,C\} \sqsubseteq \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\}.
\end{aligned}$$

Remark 5.5.1. *According to* [134], *the sets*

$$A\{2\}_{\mathcal{R}(B),*} = \left\{A^{(2)}_{\mathcal{R}(B),*}\right\},\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p};$$
$$A\{2\}_{*,\mathcal{N}(C)} = \left\{A^{(2)}_{*,\mathcal{N}(C)}\right\},\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p};$$
$$A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)} = \left\{A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right\},\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$$

assume fixed A,B,C, *while* $A\{2:\mathcal{R}(B),*\}$, $A\{2:*,\mathcal{N}(C)\}$ *and* $A\{2:\mathcal{R}(B),\mathcal{N}(C)\}$ *assume only fixed* A, *but variable* B *and* C.

5.5.1 Extensions of OMP inverses

The replacement of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ by $\Phi_1 := B(CAB)^{(1)}C$ in the definition $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$ of the OMP inverse leads to extensions of OMP inverses to expressions of more general form $\Phi_1AA^\dagger$. Moreover, there is a proper possibility to use the identity $\Phi_1AA^\dagger = B(CAA^\dagger AB)^{(1)}CAA^\dagger$, which enables direct application of the Urquhart's representation under proper rank conditions.

Theorem 5.5.1. *An arbitrary element* $X \in A_{\Gamma_\dagger}$ *from the* Φ_1-OMP *class of* $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ *defined by arbitrary but fixed* $\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ *satisfies*
(1) $X \in A\{2\}_{\mathcal{R}(B),*} \iff \mho_{CAB,B}$;
(2) $X \in A\{2\}_{*,\mathcal{N}(CAA^\dagger)} \iff \mho^C_{CAB,CA}$;
(3) $X \in A^{(2)}_{*,\mathcal{N}(C)}AA^\dagger = A\{2\}_{*,\mathcal{N}(CAA^\dagger)} \iff \mho_{CAB,C}$;
(4) $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)} \iff \mho^C_{CAB,B,CA}$;
(5) $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger \iff \mho_{CAB,B,C}$;
(6) $X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)} = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger \iff \mho_{CAB,B,C,A}$.

Proof. The proof is based on the identity

$$X := \Phi_1AA^\dagger = B(CAA^\dagger AB)^{(1)}CAA^\dagger$$

and the Urquhart representation. Moreover, the rank relationship $\mho_{CAA^\dagger,CA}$ is applied. ☐

It is useful to deduce corresponding set identities from Theorem 5.5.1. Set identities discovered in Theorem 5.5.2 show that Φ_1-OMP inverses are particular outer inverses under certain rank assumptions.

Theorem 5.5.2. *The* Φ_1-OMP *class of* $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ *satisfies the following set identities:*
(1) $A_{\Omega_\dagger} = A_{\Gamma_\dagger} \cap \Theta_{CAB,C,B}$;
(2) $A\{2:\mathcal{R}(B),*\} = A_{\Gamma_\dagger} \cap \Theta_{CAB,B}$;
(3) $A\{2:*,\mathcal{N}(CAA^\dagger)\} = A_{\Gamma_\dagger} \cap \Theta^C_{CAB,CA}$;
(4) $A\{2:*,\mathcal{N}(CAA^\dagger)\} = A\{2:*,\mathcal{N}(C)\}AA^\dagger = A_{\Gamma_\dagger} \cap \Theta_{CAB,C}$;
(5) $A\{2:\mathcal{R}(B),\mathcal{N}(CAA^\dagger)\} = A_{\Gamma_\dagger} \cap \Theta^C_{CAB,B,CA}$;
(6) $A\{2:\mathcal{R}(B),\mathcal{N}(CAA^\dagger)\} = A\{2:\mathcal{R}(B),\mathcal{N}(C)\}\,AA^\dagger = A_{\Gamma_\dagger} \cap \Theta_{CAB,B,C}$;
(7) $A\{1,2:\mathcal{R}(B),\mathcal{N}(C)\}AA^\dagger = A\{1,2:\mathcal{R}(B),\mathcal{N}(CAA^\dagger)\} = A_{\Gamma_\dagger} \cap \Theta_{CAB,B,C,A}$.

Proof. (1) An arbitrary OMP inverse X of A satisfies

$$\begin{aligned} X \in A_{\Omega_\dagger} &\iff \exists\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} : X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger \\ &\iff \exists\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} : X = \Phi_1AA^\dagger \bigwedge \mho_{CAB,C,B} \\ &\iff X \in \left\{\Phi_1AA^\dagger |\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,C,B}\right\} \\ &\iff X \in A_{\Gamma_\dagger} \cap \Theta_{CAB,C,B}. \end{aligned}$$

(3) On the basis of Lemma 5.5.1, an arbitrary $X \in A\{2:*,\mathcal{N}(CAA^\dagger)\}$ satisfies

$$\begin{aligned} X \in A\{2:*,\mathcal{N}(CAA^\dagger)\} &\iff \exists\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} : X = A^{(2)}_{*,\mathcal{N}(CAA^\dagger)} \\ &\iff X \in \left\{B(CAA^\dagger AB)^{(1)}CAA^\dagger |\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,CA}\right\} \\ &\iff X \in \left\{\Phi_1AA^\dagger |\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,CA}\right\} \\ &\iff X \in \left\{\Phi_1AA^\dagger |\ \{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\} \cap \Theta_{CAB,CA} \\ &\iff X \in A_{\Gamma_\dagger} \cap \Theta_{CAB,CA}. \end{aligned}$$

Another statements can be verified analogously. $\square$

Necessary and sufficient conditions for $X \in A^{(2)}_{\mathcal{R}(B),*}$ are considered in in [132, Theorem 3]. According to [132, Theorem 5], [133] and $\mathcal{R}(CAA^\dagger) = \mathcal{R}(CA)$, it follows that $A^{(2)}_{*,\mathcal{N}(CAA^\dagger)}$ always exists.

Corollary 5.5.1 can be verified using these results.

Corollary 5.5.1. *The set $A\{2\}_{*,\mathcal{N}(CAA^\dagger)}$ is characterized in the following manner*

$$\begin{aligned}\emptyset \neq A\{2\}_{*,\mathcal{N}(CAA^\dagger)} &= \left\{(CA)^{(1)}CAA^\dagger + (I-(CA)^{(1)}CA)UCAA^\dagger |\ U \in \mathbb{C}^{n\times q} \bigwedge \mho_{CA,C}\right\}\\ &= (CA)\{1\}CAA^\dagger \cap \Theta_{CA,C}\\ &= B(CAB)\{1\}CAA^\dagger \cap \Theta_{CAB,CA}.\end{aligned}$$

A careful analysis of Theorem 5.5.1 reveals that an arbitrary $X \in A_{\Gamma_\dagger}$ (for arbitrary but fixed $\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$) belongs to two general classes of generalized inverses, namely $X \in A\{2\}_{\mathcal{R}(B),*}$ or $X \in A\{2\}_{*,\mathcal{N}(CAA^\dagger)}$. According to the results presented in Theorem 5.5.1, we state Algorithm 5.5.1 for calculating Φ_1-OMP inverses. The underlying fact is that the matrix equation $BUCAB = B$ is solvable under the conditions $\mho_{CAB,B}$, while $CABUCA = CA$ is solvable in the case $\mho_{CAB,CA}$. In both cases, $U \subseteq (CAB)\{1\}$ and $BUCAA^\dagger$ is desired output. In this way, Algorithm 5.5.1 represents a continuation of main idea used in the computational procedures developed in [132, 133].

Algorithm 5.5.1 Computing Φ_1-OMP inverses.

Input: $\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$.
If $\mho_{CAB,B}$ **Then**
Solve $BUCAB = B$.
Compute $X := BUCAA^\dagger \subseteq A\{2\}_{\mathcal{R}(B),*}$.
End If
If $\mho_{CAB,CA}$ **Then**
Solve $CABUCA = CA$.
Compute $X := BUCAA^\dagger \subseteq A\{2\}_{*,\mathcal{N}(CAA^\dagger)}$.
End If
Return X.

The $\{2\}$-inverse $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$ exist in the case of the both restrictions $\mho_{CAB,B}$ and $\mho_{CAB,CA}$. So, it can be characterized by the results from [132, Theorem 6].

Corollary 5.5.2. **(a)** *The next characterizations of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$ are mutually equivalent to one another, where $\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$:*

(i) *$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$ exists;*

(ii) *$BUCAB = B$ and $CABUCA = CA$ for some $U \in \mathbb{C}(\mathbf{x})^{p\times q}$;*

(iii) *$BUCAB = B$ and $CABVCA = CA$ for some $U, V \in \mathbb{C}(\mathbf{x})^{p\times q}$;*

(iv) *$BUAB = B$, $CAVCA = CA$ and $BU = VCAA^\dagger$ for some $U \in \mathbb{C}^{p\times m}$ and $V \in \mathbb{C}(\mathbf{x})^{n\times q}$;*

(v) *$VCAB = B$ and $CABU = CAA^\dagger$ for some $U \in \mathbb{C}^{p\times m}$ and $V \in \mathbb{C}(\mathbf{x})^{n\times q}$;*

(vi) *$\mathcal{R}(CAB) = \mathcal{R}(CA)$ and $\mathcal{N}(CAB) = \mathcal{N}(B)$;*

(vii) *$\Phi_1 AB = B$ and $CA\Phi_1 A = CA$.*

(b) *If the claims in* **(a)** *are true, then*

$$\begin{aligned}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)} &= \Phi_1 AA^\dagger\\ &= BUCAA^\dagger,\end{aligned} \tag{5.10}$$

for arbitrary $U \in \mathbb{C}^{p\times q}$ satisfying $BUCAB = B$ and $CABUCA = CA$.

In addition, [135, Theorem 2.1] implies the following characterizations and the existence conditions of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$.

Corollary 5.5.3. **(a)** *The following assertions are mutually equivalent for $\{A,B,C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$:*

(i) $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^{\dagger})}$ *exists;*

(ii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = B, \quad CAX = CAA^{\dagger}, \quad \mathcal{R}(X) \subseteq \mathcal{R}(B), \quad \mathcal{N}(CAA^{\dagger}) \subseteq \mathcal{N}(X);$$

(iii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = B, \quad CAX = CAA^{\dagger}, \quad X = BB^{(1)}X = X(CAA^{\dagger})^{(1)}CAA^{\dagger}.$$

(iv) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = B, \quad CAX = CAA^{\dagger}, \quad X \in B\mathbb{C}^{p\times n}X \cap X\mathbb{C}^{m\times q}CAA^{\dagger};$$

(v) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = B, \quad CAX = CAA^{\dagger}, \quad X = BU = VCAA^{\dagger},$$

for some $U \in \mathbb{C}^{p\times m}$, $V \in \mathbb{C}^{n\times q}$;

(vi) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = B, \quad CAX = CAA^{\dagger}, \quad X = BUCAA^{\dagger}, \quad \text{for some } U \in \mathbb{C}^{p\times q}.$$

(b) *If any of the statements* (ii)-(vi) *holds, then* $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^{\dagger})}$.

In general, $\Phi_1 AA^{\dagger}$ is not an outer inverse of A. One of such cases is $\mho^{B,CA}_{CAB}$. But if we replace $(CAB)^{(1)}$ with some $(CAB)^{(2)}$ (or only $(CAB)^{(1,2)}$) then $\Phi_2 AA^{\dagger}$, $\Phi_2 := B(CAB)^{(2)}C$, is an outer inverse of A.

Theorem 5.5.3. **(a)** *The following statements are equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$*:*

(i) $X := \Phi_2 AA^{\dagger} \in A_{\Gamma_{\dagger}\{2\}}$*;*

(ii) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAX = X, \quad AX = A\Phi_2 AA^{\dagger}, \quad XA = \Phi_2 A;$$

(iii) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XA\Phi_2 AA^{\dagger} = X, \quad XA = \Phi_2 A;$$

(iv) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XA\Phi_2 AA^{\dagger} = X, \quad XAA^{*} = \Phi_2 AA^{*};$$

(v) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XA\Phi_2 AA^{\dagger} = X, \quad XAA^{\dagger} = \Phi_2 AA^{\dagger};$$

(vi) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$\Phi_2 AX = X, \quad AX = A\Phi_2 AA^{\dagger};$$

(vii) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAX = X, \quad AXA = A\Phi_2 A, \quad AX = A\Phi_2 AA^{\dagger}, \quad XA = \Phi_2 A;$$

(viii) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAA^{\dagger} = X, \quad XA = \Phi_2 A;$$

(ix) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAA^{\dagger} = X, \quad XAA^{*} = \Phi_2 AA^{*};$$

(x) *there is* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$BB^{(1)}X = X, \quad B^{(1)}X = B^{(1)}\Phi_2 AA^{\dagger}.$$

(b) *If an arbitrary of the statements* (ii)-(x) *is valid, then* $X = \Phi_2 AA^{\dagger}$ *is unique.*

Proof. (i) $\Rightarrow$ (ii): For $X = \Phi_2 AA^\dagger$, notice that $AX = A\Phi_2 AA^\dagger$ and $XA = \Phi_2 A$, which gives

$$XAX = B\left((CAB)^{(2)}CAB(CAB)^{(2)}\right)CAA^\dagger = \Phi_2 AA^\dagger = X.$$

(ii) $\Rightarrow$ (iii): We observe that the assumptions $XAX = X$ and $AX = A\Phi_2 AA^\dagger$ initiate $X = X(AX) = XA\Phi_2 AA^\dagger$.

(iii) $\Rightarrow$ (iv): Multiplying $XA = \Phi_2 A$ by A^* from the right hand side, it follows $XAA^* = \Phi_2 AA^*$.

(iv) $\Rightarrow$ (v): Since $XAA^* = \Phi_2 AA^*$, it follows

$$XAA^\dagger = (XAA^*)(A^\dagger)^* A^\dagger = \Phi_2 AA^*(A^\dagger)^* A^\dagger = \Phi_2 AA^\dagger.$$

(v) $\Rightarrow$ (i): Applying $XA\Phi_2 CAA^\dagger = X$ and $XAA^\dagger = \Phi_2 AA^\dagger$, we get

$$X = (XAA^\dagger)A\Phi_2 AA^\dagger = \Phi_2 AA^\dagger A\Phi_2 AA^\dagger = \Phi_2 AA^\dagger.$$

The remaining parts of the proof are verified using a similar approach. $\square$

Recall that, by [8], $X \in A\{2\}$ if and only if it is of the form $X = (EAF)^\dagger$, where E and F are suitable Hermitian idempotents.

Lemma 5.5.2. *For $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$, an arbitrary $X := \Phi_2 AA^\dagger \in A_{\Gamma_\dagger}^{\{2\}}$ satisfies*

$$X = B(ECABF)^\dagger CAA^\dagger = BF(ECABF)^\dagger ECAA^\dagger, \tag{5.11}$$

where E and F are Hermitian idempotents.

Proof. It is clear by [8] and based on $(CAB)^{(2)} = (ECABF)^\dagger = F(ECABF)^\dagger = (ECABF)^\dagger E$. $\square$

The next results are obtained by applying Theorem 5.5.1 and Lemma 5.5.2.

Corollary 5.5.4. *An arbitrary element $X \in A_{\Gamma_\dagger}^{\{2\}}$ defined by arbitrary but fixed $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ and represented by* (5.11) *satisfies*
(1) $X \in A\{2\}_{\mathcal{R}(BF),*} \Longleftrightarrow \mho_{ECABF,BF}$;
(2) $X \in A\{2\}_{*,\mathcal{N}(ECAA^\dagger)} \Longleftrightarrow \mho_{ECABF,ECA}^{EC}$;
(3) $X \in A^{(2)}_{*,\mathcal{N}(EC)}AA^\dagger = A\{2\}_{*,\mathcal{N}(ECAA^\dagger)} \Longleftrightarrow \mho_{ECABF,EC}$;
(4) $X = A^{(2)}_{\mathcal{R}(BF),\mathcal{N}(ECAA^\dagger)} \Longleftrightarrow \mho_{ECABF,BF,ECA}^{EC}$;
(5) $X = A^{(2)}_{\mathcal{R}(BF),\mathcal{N}(ECAA^\dagger)} = A^{(2)}_{\mathcal{R}(BF),\mathcal{N}(EC)}AA^\dagger \Longleftrightarrow \mho_{ECABF,BF,EC}$;
(6) $X = A^{(1,2)}_{\mathcal{R}(BF),\mathcal{N}(ECAA^\dagger)} = A^{(1,2)}_{\mathcal{R}(BF),\mathcal{N}(EC)}AA^\dagger \Longleftrightarrow \mho_{ECABF,BF,EC,A}$,
where E and F are Hermitian idempotents involved in (5.11).

5.5.2 Extensions of MPO inverses

Also, it is possible to consider extension of MPO inverses of more general form $A^\dagger A\Phi_1$. Under the rank conditions $\mho_{CAB,B,C}$, it follows $A^\dagger A\Phi_1 = A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. Moreover, there is a proper possibility to use the identity $A^\dagger A\Phi_1 = A^\dagger AB(CAA^\dagger AB)^{(1)}C$, which enables direct application of the Urquhart's representation under proper rank conditions $\mho_{CAB,AB}^{B}$.

Theorem 5.5.4. *An arbitrary element $X \in A_{\dagger\Gamma}$ from the Φ_1-MPO class of $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ defined by arbitrary but fixed $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ satisfies*
(1) $X \in A\{2\}_{*,\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,C}$;
(2) $X \in A\{2\}_{\mathcal{R}(A^\dagger AB),*} \Longleftrightarrow \mho_{CAB,AB}^{B}$;
(3) $X \in A^\dagger AA^{(2)}_{\mathcal{R}(B),*} = A\{2\}_{\mathcal{R}(A^\dagger AB),*} \Longleftrightarrow \mho_{CAB,B}$;
(4) $X = A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,AB,C}^{B}$;
(5) $X = A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)} = A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,C,B}$;
(6) $X = A^{(1,2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)} = A^\dagger AA^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} \Longleftrightarrow \mho_{CAB,C,B,A}$.

Proof. The proof follows from

$$A^\dagger AB(CAB)^{(1)}C = A^\dagger AB(CAA^\dagger AB)^{(1)}C$$

in conjunction with the Urquhart formula and the rank identity $\mho_{A^\dagger AB,AB}$. $\square$

Theorem 5.5.5 reveals that the Φ_1-MPO class is a subset of outer inverses.

Theorem 5.5.5. *The* Φ_1-MPO *class of* $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ *satisfies the following set identities:*
(1) $A_{\dagger}\Omega = A_{\dagger}\Gamma \cap \Theta_{CAB,C,B}$;
(2) $A\{2 : *, \mathcal{N}(C)\} = A_{\dagger}\Gamma \cap \Theta_{CAB,C}$;
(3) $A\{2 : \mathcal{R}(A^\dagger AB), *\} = A_{\dagger}\Gamma \cap \Theta^{B}_{CAB,AB}$;
(4) $A^\dagger A\,A\{2 : \mathcal{R}(B), *\} = A\{2 : \mathcal{R}(A^\dagger AB), *\} = A_{\dagger}\Gamma \cap \Theta_{CAB,B}$;
(5) $A\{2 : \mathcal{R}(A^\dagger AB), \mathcal{N}(C)\} = A_{\dagger}\Gamma \cap \Theta^{B}_{CAB,AB,C}$;
(6) $A\{2 : \mathcal{R}(A^\dagger AB), \mathcal{N}(C)\} = A^\dagger A \cdot A\{2 : \mathcal{R}(B), \mathcal{N}(C)\} = A_{\dagger}\Gamma \cap \Theta_{CAB,C,B}$;
(7) $A^\dagger A\,A\{1, 2 : \mathcal{R}(B), \mathcal{N}(C)\} = A\{1, 2 : \mathcal{R}(B), \mathcal{N}(CAA^\dagger)\} = A_{\dagger}\Gamma \cap \Theta_{CAB,C,B,A}$.

Proof. (3) An arbitrary $X \in A\{2 : \mathcal{R}(A^\dagger AB), *\}$ satisfies

$$\begin{aligned}
X \in A\{2 : \mathcal{R}(A^\dagger AB), *\} &\iff \exists \{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} : X = A^{(2)}_{\mathcal{R}(A^\dagger AB),*}\\
&\iff X \in \left\{A^\dagger AB(CAB)^{(1)}C \mid \{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p} \bigwedge \mho_{CAB,AB}\right\}\\
&\iff X \in \left\{A^\dagger A\Phi_1 \mid \{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}\right\} \cap \Theta_{CAB,AB}\\
&\iff X \in A_{\dagger}\Gamma \cap \Theta_{CAB,AB}.
\end{aligned}$$

Verification of remaining statements is analogous. □

By [132, Theorem 3] and $\mathcal{N}(AB) = \mathcal{N}(A^\dagger AB)$, we observe that the set $A\{2\}_{\mathcal{R}(A^\dagger AB),*}$ is not empty. The verification of Corollary 5.5.5 is based on these results.

Corollary 5.5.5. *The set* $A\{2\}_{\mathcal{R}(A^\dagger AB),*}$ *is not empty and satisfies*

$$\begin{aligned}
\emptyset \neq A\{2\}_{\mathcal{R}(A^\dagger AB),*} &= \left\{A^\dagger AB(AB)^{(1)} + A^\dagger ABU(I - AB(AB)^{(1)}) : U \in \mathbb{C}^{p\times m} \bigwedge \mho_{AB,B}\right\}\\
&= A^\dagger AB(AB)\{1\} \cap \Theta_{AB,B}\\
&= A^\dagger AB(CAB)\{1\}C \cap \Theta_{CAB,AB}.
\end{aligned}$$

A detailed analysis of Theorem 5.5.4 reveals that $X \in A_{\dagger}\Gamma$ for arbitrary but fixed $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ belongs to two general classes of generalized inverses, precisely $X \in A\{2\}_{\mathcal{R}(A^\dagger AB),*}$ or $X \in A\{2\}_{*,\mathcal{N}(C)}$. Algorithm 5.5.2 for computing Φ_1-MPO inverses is stated in view of the results presented in Theorem 5.5.4. The motivation is the fact that the matrix equation $ABUCAB = AB$ is solvable under the conditions $\mho_{CAB,AB}$ and $CABUC = C$ is solvable in the case $\mho_{CAB,A}$. In both cases, $U \in (CAB)\{1\}$ and $BUCAA^\dagger$ is desired output. In this way, Algorithm 5.5.2 is a continuation of computational procedures developed in [132, 133].

Algorithm 5.5.2 Computing Φ_1-MPO inverses.

Input: $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$.
1: **If** $\mho_{CAB,AB}$ **Then**
2: Solve $ABUCAB = AB$.
3: Compute $X := A^\dagger ABUC \subseteq A\{2\}_{\mathcal{R}(A^\dagger AB),*}$.
End If
4: **If** $\mho_{CAB,C}$ **Then**
5: Solve $CABUC = C$.
6: Compute $X := A^\dagger ABUC \subseteq A\{2\}_{*,\mathcal{N}(C)}$.
End If
7: Return X.

The existence and representations of $A^{(2)}_{*,\mathcal{N}(C)}$ can be derived in the same way as in [132, Theorem 5] and [133]. We present equivalent conditions for the existence of $A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)}$ using [132, Theorem 6].

Corollary 5.5.6. **(a)** *The next assertions are mutually equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$:

(i) $A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)}$ *exists;*
(ii) $ABUCAB = AB$ *and* $CABUC = C$ *for some* $U \in \mathbb{C}(\mathbf{x})^{p\times q}$;
(iii) $ABUCAB = AB$ *and* $CABVC = C$ *for some* $U, V \in \mathbb{C}(\mathbf{x})^{p\times q}$;
(iv) $ABUAB = AB$, $CAVC = C$ *and* $A^\dagger ABU = VC$ *for some* $U \in \mathbb{C}(\mathbf{x})^{p\times m}$ *and* $V \in \mathbb{C}(\mathbf{x})^{n\times q}$;

(v) $VCAB = A^\dagger AB$ *and* $CABU = C$ *for some* $U \in \mathbb{C}(\mathbf{x})^{p\times m}$ *and* $V \in \mathbb{C}(\mathbf{x})^{n\times q}$*;*

(vi) $\mathcal{R}(CAB) = \mathcal{R}(C)$ *and* $\mathcal{N}(CAB) = \mathcal{N}(AB)$*;*

(vii) $A\Phi_1 AB = AB$ *and* $CA\Phi_1 = C$*.*

(b) *If the conditions in* **(a)** *hold, it follows*

$$\begin{aligned} A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)} &= A^\dagger A\Phi_1 \\ &= A^\dagger ABUC, \end{aligned} \tag{5.12}$$

for arbitrary $U \in \mathbb{C}^{p\times q}$ *satisfying* $ABUCAB = AB$ *and* $CABUC = C$*.*

[135, Theorem 2.1] implies more characterizations for the existence of $A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)}$.

Corollary 5.5.7. **(a)** *The following statements are equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$*:*

(i) $A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)}$ *exists;*

(ii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^\dagger AB, \quad CAX = C, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger AB), \quad \mathcal{N}(C) \subseteq \mathcal{N}(X);$$

(iii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *satisfying*

$$XAB = BA^\dagger A, \quad CAX = C, \quad X = A^\dagger AB(A^\dagger AB)^{(1)}X = XC^{(1)}C.$$

(iv) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = BA^\dagger A, \quad CAX = C, \quad X \in A^\dagger AB\mathbb{C}^{p\times n}X \cap X\mathbb{C}^{m\times q}C;$$

(v) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^\dagger AB, \quad CAX = C, \quad X = A^\dagger ABU = VC,$$

for some $U \in \mathbb{C}^{p\times m}$*,* $V \in \mathbb{C}^{n\times q}$*;*

(vi) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^\dagger AB, \quad CAX = C, \quad X = A^\dagger ABUC, \quad \textit{for some } U \in \mathbb{C}^{p\times q}.$$

(b) *If the statements* (ii)-(vi) *are valid, then* $X = A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(C)}$.

Now, we replace $(CAB)^{(1)}$ with $(CAB)^{(2)}$ in the expression $A^\dagger A\Phi_1$ and consider characterizations for $X = A^\dagger A\Phi_2$ to hold. Theorem 5.5.6 can be proved as Theorem 5.5.3.

Theorem 5.5.6. **(a)** *The next statements are equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$*:*

(i) $X := A^\dagger A\Phi_2 \in A_{\dagger\Gamma\{2\}}$*;*

(ii) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to the matrix system*

$$XAX = X, \quad AX = A\Phi_2, \quad XA = A^\dagger A\Phi_2 A;$$

(iii) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$A^\dagger A\Phi_2 AX = X, \quad AX = A\Phi_2;$$

(iv) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$A^\dagger A\Phi_2 CAX = X, \quad A^*AX = A^*A\Phi_2;$$

(v) *there is a solution* $X \in \mathbb{C}^{n\times m}$ *to*

$$A^\dagger A\Phi_2 AX = X, \quad A^\dagger AX = A^\dagger A\Phi_2;$$

(vi) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$XA\Phi_2 = X, \quad XA = A^\dagger A\Phi_2 A;$$

(vii) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$XAX = X, \quad AXA = A\Phi_2 A, \quad AX = A\Phi_2, \quad XA = A^\dagger A\Phi_2 A;$$

(viii) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$A^\dagger AX = X, \quad AX = A\Phi_2;$$

(ix) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$A^\dagger AX = X, \quad A^*AX = A^*A\Phi_2;$$

(x) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$XC^{(1)}C = X, \quad XC^{(1)} = A^\dagger A\Phi_2 C^{(1)}.$$

(b) *If any of the statements* (ii)-(x) *is confirmed, then* $X = A^\dagger A\Phi_2$ *is unique.*

To consider the relation between $X \in A_{\dagger\Gamma\{2\}}$ and $X \in A_{\dagger\Gamma}$, we need the next auxiliary Lemma 5.5.3.

Lemma 5.5.3. *An arbitrary* $X := A^\dagger A\Phi_2 \in A_{\dagger\Gamma\{2\}}$ *satisfies*

$$X = A^\dagger AB(ECABF)^\dagger C = A^\dagger ABF(ECABF)^\dagger EC, \tag{5.13}$$

where E *and* F *are Hermitian idempotents.*

By Theorem 5.5.1 and Lemma 5.5.3, we verify the following characterizations of $X \in A_{\dagger\Gamma\{2\}}$.

Corollary 5.5.8. *An arbitrary element* $X \in A_{\dagger\Gamma\{2\}}$ *defined by arbitrary but fixed* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ *and represented by* (5.13) *satisfies*
(1) $X \in A\{2\}_{*,\mathcal{N}(EC)} \iff \mho_{ECABF,EC}$;
(2) $X \in A\{2\}_{\mathcal{R}(A^\dagger ABF),*} \iff \mho^{BF}_{ECABF,ABF}$;
(3) $X \in A^\dagger AA^{(2)}_{\mathcal{R}(BF),*} = A\{2\}_{\mathcal{R}(A^\dagger ABF),*} \iff \mho_{ECABF,BF}$;
(4) $X = A^{(2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(EC)} \iff \mho^{BF}_{CAB,AB,C}$;
(5) $X = A^{(2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(EC)} = A^\dagger AA^{(2)}_{\mathcal{R}(BF),\mathcal{N}(EC)} \iff \mho_{ECABF,EC,BF}$;
(6) $X = A^{(1,2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(EC)} = A^\dagger AA^{(1,2)}_{\mathcal{R}(BF),\mathcal{N}(EC)} \iff \mho_{ECABF,EC,BF,A}$,
where E *and* F *are Hermitian idempotents involved in* (5.11).

5.5.3 Extensions of MPOMP inverses

MPOMP inverses can be extended in more general form $A^\dagger A\Phi_1 AA^\dagger$. Moreover, there is the possibility to use the identity $A^\dagger A\Phi_1 AA^\dagger = A^\dagger AB(CAA^\dagger AA^\dagger AB)^{(1)}CAA^\dagger$, which enables direct application of the Urquhart's representation under proper rank conditions.

Theorem 5.5.7. *An arbitrary element* $X \in A_{\dagger\Gamma_\dagger}$ *from the* Φ_1-MPOMP *class of* $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ *determined by arbitrary but fixed* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ *satisfies*
(1) $X \in A\{2\}_{\mathcal{R}(A^\dagger AB),*} \iff \mho^{B}_{CAB,AB}$;
(2) $X \in A\{2\}_{*,\mathcal{N}(CAA^\dagger)} \iff \mho^{C}_{CAB,CA}$;
(3) $X = A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(CAA^\dagger)} \iff \mho^{B}_{CAB,AB,CA} \wedge \mho^{C}_{CAB,AB,CA}$;
(4) $X = A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(CAA^\dagger)} = A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger \iff \mho_{CAB,B,C}$;
(5) $X = A^{(1,2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(CAA^\dagger)} = A^\dagger AA^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger \iff \mho_{CAB,B,C,A}$;
(6) $X = A^{(1,2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(CAA^\dagger)} \iff \mho_{CAB,AB,CA,A}$.

Proof. The proof follows from

$$A^\dagger A\,B(CAB)^{(1)}C\,AA^\dagger = A^\dagger AB(CAA^\dagger\,A\,A^\dagger AB)^{(1)}CAA^\dagger$$

and the Urquhart representation in conjunction with assumptions $\mho_{A^\dagger AB,AB}$ and $\mho_{CAA^\dagger,CA}$. ☐

Corollary 5.5.9. *The* Φ_1-MPOMP *class of* $A \in \mathbb{C}(\mathbf{x})^{m\times n}$ *satisfies the next set identities:*
(1) $A_{\dagger\Omega_\dagger} = A_{\dagger\Gamma_\dagger} \cap \Theta_{CAB,C,B}$;
(2) $A\{2 : \mathcal{R}(A^\dagger AB), *\} = A_{\dagger\Gamma_\dagger} \cap \Theta^{B}_{CAB,AB}$;
(3) $A\{2 : *, \mathcal{N}(CAA^\dagger)\} = A_{\dagger\Gamma_\dagger} \cap \Theta^{C}_{CAB,CA}$;
(4) $A\{2 : \mathcal{R}(A^\dagger AB), \mathcal{N}(CAA^\dagger)\} = A_{\dagger\Gamma_\dagger} \cap \Theta^{B}_{CAB,AB,CA} \cap \Theta^{C}_{CAB,AB,CA}$;
(5) $A\{2 : \mathcal{R}(A^\dagger AB), \mathcal{N}(CAA^\dagger)\} = A^\dagger A \cdot A\{2 : \mathcal{R}(B), \mathcal{N}(C)\}AA^\dagger = A_{\dagger\Gamma_\dagger} \cap \Theta_{CAB,B,C}$;
(6) $A\{1, 2 : \mathcal{R}(A^\dagger AB), \mathcal{N}(CAA^\dagger)\} = A^\dagger A \cdot A\{1, 2 : \mathcal{R}(B), \mathcal{N}(C)\}AA^\dagger = A_{\dagger\Gamma_\dagger} \cap \Theta_{CAB,B,C,A}$;
(7) $A\{1, 2 : \mathcal{R}(A^\dagger AB), \mathcal{N}(CAA^\dagger)\} = A_{\dagger\Gamma_\dagger} \cap \Theta_{CAB,AB,CA,A}$.

Investigation of Theorem 5.5.7 reveals that $X \in_{\dagger}\Gamma_{\dagger}$ belongs to two general classes of generalized inverses, namely $X \in A\{2\}_{\mathcal{R}(A^{\dagger}AB),*}$ or $X \in A\{2\}_{*,\mathcal{N}(CAA^{\dagger})}$. Algorithm 5.5.3 for computing Φ_1-MPOMP inverses is stated on the basis of the results presented in Theorem 5.5.7 and previous analysis.

Algorithm 5.5.3 Computing Φ_1-MPOMP inverses.

Input: $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$.
If $\mho_{CAB,AB}$ **Then**
Solve $ABUCAB = AB$.
Compute $X := A^{\dagger}ABUCAA^{\dagger} \subseteq A\{2\}_{\mathcal{R}(A^{\dagger}AB),*}$.
End If
If $\mho_{CAB,CA}$ **Then**
Solve $CABUCA = CA$.
Compute $X := A^{\dagger}ABUCAA^{\dagger} \subseteq A\{2\}_{*,\mathcal{N}(CAA^{\dagger})}$.
End If
Return X.

We consider the existence and representations of $A^{(2)}_{\mathcal{R}(A^{\dagger}AB),\mathcal{N}(CAA^{\dagger})}$ in the next result by [132, Theorem 6].

Corollary 5.5.10. **(a)** *The following statements are mutually equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$*:*

(i) $A^{(2)}_{\mathcal{R}(A^{\dagger}AB),\mathcal{N}(CAA^{\dagger})}$ *exists;*

(ii) $ABUCAB = AB$ *and* $CABUCA = CA$ *for some* $U \in \mathbb{C}(\mathbf{x})^{p\times q}$*;*

(iii) $ABUCAB = AB$ *and* $CABVCA = CA$ *for certain* $U, V \in \mathbb{C}(\mathbf{x})^{p\times q}$*;*

(iv) $ABUAB = AB$*,* $CAVCA = CA$ *and* $A^{\dagger}ABU = VCAA^{\dagger}$ *for some* $U \in \mathbb{C}(\mathbf{x})^{p\times m}$ *and* $V \in \mathbb{C}(\mathbf{x})^{n\times q}$*;*

(v) $VCAB = A^{\dagger}AB$ *and* $CABU = CAA^{\dagger}$ *for some* $U \in \mathbb{C}(\mathbf{x})^{p\times m}$ *and* $V \in \mathbb{C}(\mathbf{x})^{n\times q}$*;*

(vi) $\mathcal{R}(CAB) = \mathcal{R}(CA)$ *and* $\mathcal{N}(CAB) = \mathcal{N}(AB)$*;*

(vii) $A\Phi_1 AB = AB$ *and* $CA\Phi_1 A = CA$*.*

(b) *If the conditions in* **(a)** *are fulfilled, then*

$$\begin{aligned} A^{(2)}_{\mathcal{R}(A^{\dagger}AB),\mathcal{N}(C)} &= A^{\dagger}A\Phi_1 AA^{\dagger} \\ &= A^{\dagger}ABUCAA^{\dagger}, \end{aligned} \tag{5.14}$$

for arbitrary $U \in \mathbb{C}(\mathbf{x})^{p\times q}$ *satisfying* $ABUCAB = AB$ *and* $CABUCA = CA$*.*

By [135, Theorem 2.1], we obtain the next result related to $A^{(2)}_{\mathcal{R}(A^{\dagger}AB),\mathcal{N}(CAA^{\dagger})}$.

Corollary 5.5.11. **(a)** *The subsequent claims are equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$*:*

(i) $A^{(2)}_{\mathcal{R}(A^{\dagger}AB),\mathcal{N}(CAA^{\dagger})}$ *exists;*

(ii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^{\dagger}AB, \quad CAX = CAA^{\dagger}, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^{\dagger}AB), \quad \mathcal{N}(CAA^{\dagger}) \subseteq \mathcal{N}(X);$$

(iii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^{\dagger}AB,\ CAX = CAA^{\dagger},\ X = A^{\dagger}AB(A^{\dagger}AB)^{(1)}X = X(CAA^{\dagger})^{(1)}CAA^{\dagger}.$$

(iv) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^{\dagger}AB, \quad CAX = CAA^{\dagger}, \quad X \in A^{\dagger}AB\mathbb{C}^{p\times n}X \cap X\mathbb{C}^{m\times q}CAA^{\dagger};$$

(vi) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^{\dagger}AB, \quad CAX = CAA^{\dagger}, \quad X = A^{\dagger}ABU = VCAA^{\dagger},$$

for some $U \in \mathbb{C}^{p\times m}$*,* $V \in \mathbb{C}^{n\times q}$*;*

(vii) *there exists* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *which satisfies*

$$XAB = A^{\dagger}AB, \quad CAX = CAA^{\dagger}, \quad X = A^{\dagger}ABUCAA^{\dagger}, \quad \textit{for some } U \in \mathbb{C}^{p\times q}.$$

(b) *If any of the statements* (ii)-(vi) *is valid, then* $X = A^{(2)}_{\mathcal{R}(A^\dagger AB),\mathcal{N}(CAA^\dagger)}$.

As Theorem 5.5.3, we characterize $X = A^\dagger A\Phi_2 AA^\dagger$ in Theorem 5.5.8.

Theorem 5.5.8. **(a)** *The subsequent assertions are mutually equivalent for* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ *and* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$*:*

(i) $X := A^\dagger A\Phi_2 AA^\dagger \in A_{{}_\dagger\Gamma_\dagger\{2\}}$;

(ii) *there is a solution* X *to*

$$XAX = X, \quad AX = A\Phi_2 AA^\dagger, \quad XA = A^\dagger A\Phi_2 A;$$

(iii) *there is a solution* X *to*

$$XA\Phi_2 AA^\dagger = X, \quad XA = A^\dagger A\Phi_2 A;$$

(iv) *there is a solution* X *to*

$$XA\Phi_2 AA^\dagger = X, \quad XAA^* = A^\dagger A\Phi_2 AA^*;$$

(v) *there is a solution* X *to*

$$XA\Phi_2 AA^\dagger = X, \quad XAA^\dagger = A^\dagger A\Phi_2 AA^\dagger;$$

(vi) *there is a solution* X *to*

$$A^\dagger A\Phi_2 AX = X, \quad AX = A\Phi_2 AA^\dagger;$$

(vii) *there is a solution* X *to*

$$XAX = X, \quad AXA = A\Phi_2 A, \quad AX = A\Phi_2 AA^\dagger, \quad XA = A^\dagger A\Phi_2 A;$$

(viii) *there is a solution* $X \in \mathbb{C}(\mathbf{x})^{n\times m}$ *to*

$$XAA^\dagger = X, \quad XA = A^\dagger A\Phi_2 A;$$

(ix) *there is a solution* X *to*

$$XAA^\dagger = X, \quad XAA^* = A^\dagger A\Phi_2 AA^*;$$

(x) *there is a solution* X *to*

$$A^\dagger AX = X, \quad AX = A\Phi_2 AA^\dagger;$$

(xi) *there is a solution* X *to*

$$A^\dagger AX = X, \quad A^*AX = A^*A\Phi_2 AA^\dagger;$$

(xii) *there is a solution* X *to*

$$A^\dagger A\Phi_2 AX = X, \quad A^*AX = A^*A\Phi_2 AA^\dagger;$$

(xiii) *there is a solution* X *to*

$$A^\dagger A\Phi_2 AX = X, \quad A^\dagger AX = A^\dagger A\Phi_2 AA^\dagger.$$

(b) *If the statements* (ii)-(xiii) *hold, then* $X = A^\dagger A\Phi_2 AA^\dagger$ *is unique.*

Now, we investigate conditions which ensure that $X \in A_{{}_\dagger\Gamma_\dagger\{2\}}$ becomes $X \in A_{{}_\dagger\Gamma_\dagger}$.

Lemma 5.5.4. *An arbitrary* $X := A^\dagger A\Phi_2 AA^\dagger \in A_{{}_\dagger\Gamma_\dagger\{2\}}$ *satisfies*

$$X = A^\dagger AB(ECABF)^\dagger CAA^\dagger = A^\dagger ABF(ECABF)^\dagger ECAA^\dagger, \tag{5.15}$$

where E *and* F *are Hermitian idempotents.*

Theorem 5.5.7 and Lemma 5.5.4 imply the next corollary.

Table 5.1: Proposed generalizations of composite inverses.

Composite outer inverses		Generalizations	
Name	Definition	Φ_1-composite inverses	Φ_2-composite inverses
OMP	$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$	$\Phi_1 AA^\dagger$	$\Phi_2 AA^\dagger$
MPO	$A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$	$A^\dagger A\Phi_1$	$A^\dagger A\Phi_2$
MPOMP	$A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$	$A^\dagger A\Phi_1 AA^\dagger$	$A^\dagger A\Phi_2 AA^\dagger$

Corollary 5.5.12. *An arbitrary element* $X \in A_{{}^\dagger\Gamma_{\dagger}\{2\}}$ *defined by arbitrary but fixed* $\{A, B, C\} \in \mathbb{C}(\mathbf{x})_{q,m,n,p}$ *and represented by* (5.15) *satisfies*
(1) $X \in A\{2\}_{\mathcal{R}(A^\dagger ABF),*} \iff \mho^{BF}_{ECABF,ABF}$;
(2) $X \in A\{2\}_{*,\mathcal{N}(ECAA^\dagger)} \iff \mho^{EC}_{ECABF,ECA}$;
(3) $X = A^{(2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(ECAA^\dagger)} \iff \mho^{BF}_{ECABF,ABF,ECA} \wedge \mho^{EC}_{ECABF,ABF,ECA}$;
(4) $X = A^{(2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(ECAA^\dagger)} = A^\dagger AA^{(2)}_{\mathcal{R}(BF),\mathcal{N}(EC)}AA^\dagger \iff \mho_{ECABF,BF,EC}$;
(5) $X = A^{(1,2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(ECAA^\dagger)} = A^\dagger AA^{(1,2)}_{\mathcal{R}(BF),\mathcal{N}(EC)}AA^\dagger \iff \mho_{ECABF,BF,EC,A}$;
(6) $X = A^{(1,2)}_{\mathcal{R}(A^\dagger ABF),\mathcal{N}(ECAA^\dagger)} \iff \mho_{ECABF,ABF,ECA,A}$.

Considered extensions of composite outer inverses are illustrated in Table 5.1.

Remark 5.5.2. (a) *Extensions of* DMP, MPD, CMP, MPCEP, *CEPMP *can be defined analogically to the general approach for* Φ_1-OMP, Φ_1-MPO *and* Φ_1-MPOMP *classes.*

(b) *It is important to mention that the solutions U of the matrix equations considered in Algorithms 5.5.1,5.5.2 and 5.5.3 are given in general symbolic form, which implies $U \subseteq (CAB)\{1\}$.*

(c) *For square matrices, several particular cases of Φ_1-composite outer inverses are summarized in Table 5.2.*

Table 5.2: Special cases of Φ_1-composite outer inverses.

Φ_1-composite outer inverses	Restrictions	Composite outer inverses	Ref.
$A(A^*A^2)^{(1)}A^*$	$\mathrm{ind}(A)=1$	$A^{(2),\dagger}_{\mathcal{R}(A),\mathcal{N}(A^*)} = A^{\textcircled{\#}} = A^\# AA^\dagger$	[2]
$A^*(A^2A^*)^{(1)}A$	$\mathrm{ind}(A)=1$	$A^{\dagger,(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)} = A_{\textcircled{\#}} = A^\dagger AA^\#$	[2]
$A^k(A^{2k})^{(1)}A^kA^\dagger$	$\mathrm{ind}(A)=k$	$A^{(2),\dagger}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^\dagger)} = A^{\mathrm{D},\dagger} = A^{\mathrm{D}}AA^\dagger$	[86]
$A^\dagger A^k(A^{2k})^{(1)}A^k$	$\mathrm{ind}(A)=k$	$A^{\dagger,(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(A^k)} = A^{\dagger,\mathrm{D}} = A^\dagger AA^{\mathrm{D}}$	[86]
$A^\dagger A^k(A^{2k-1})^{(1)}A^kA^\dagger$	$\mathrm{ind}(A)=k$	$A^{\dagger,(2),\dagger}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(A^kA^\dagger)} = A^{c,\dagger} = A^\dagger AA^{\mathrm{D}}AA^\dagger$	[88]
$A^\dagger A^k((A^k)^*A^k)^{(1)}(A^k)^*$	$\mathrm{ind}(A)=k$	$A^{\dagger,(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}((A^k)^*)} = A^{\dagger,\textcircled{\dagger}} = A^\dagger AA^{\textcircled{\dagger}}$	[19]
$(A^k)^*(A^k(A^k)^*)^{(1)}A^kA^\dagger$	$\mathrm{ind}(A)=k$	$A^{\dagger,(2)}_{\mathcal{R}((A^k)^*),\mathcal{N}(A^kA^\dagger)} = A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}}AA^\dagger$	[19]

5.6 Examples

In order to explain proposed generalizations, let us consider the following examples.

Example 5.6.1. Consider the input matrix

$$A = \begin{bmatrix} t+1 & t & t & t & t+1 \\ t & t-1 & t & t & t \\ t & t & t+1 & t & t \\ t & t & t & t-1 & t \\ t+1 & t & t & t & t+1 \end{bmatrix}$$

and two associate matrices

$$B=\begin{bmatrix} 2t+1 & t & t \\ t & 2t-1 & t \\ t & t & 2t+1 \\ t & t & t \\ 2t+1 & t & t \end{bmatrix}, \quad C=\begin{bmatrix} t^2+1 & t^2 & t^2 & t^2 & t^2+1 \\ t^2 & t^2-1 & t^2 & t^2 & t^2 \\ t^2 & t^2 & t^2+1 & t^2 & t^2 \end{bmatrix}.$$

It can be verified that $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C) = 3 < 4 = \operatorname{rank}(A)$, which guarantees the existence of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. Symbolic calculation in package *Mathematica* gives a common solution to the matrix equations $CABUCA = CA$, $CABUC = C$ and $BUCAB = B$ in the unique form

$U = (CAB)^{-1} =$

$$\begin{bmatrix} \frac{-36t^5-18t^4+6t^3+3t^2+1}{-63t^6-21t^5+41t^4+t^3+10t^2+12t+4} & \frac{t(42t^4+57t^3+8t^2+19t+2)}{-63t^6-21t^5+41t^4+t^3+10t^2+12t+4} & -\frac{t(6t^4+39t^3-10t^2-5t+6)}{-63t^6-21t^5+41t^4+t^3+10t^2+12t+4} \\ \frac{-63t^4-21t^3+5t^2+t}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{2(63t^4+21t^3+22t^2+10t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & -\frac{7t^2(9t^2+3t-2)}{63t^5-42t^4+t^3-2t^2-8t-4} \\ \frac{t(21t^4+30t^3-8t^2+2t+3)}{(t+1)(63t^5-42t^4+t^3-2t^2-8t-4)} & -\frac{t^2(77t^3+42t^2+11t+22)}{(t+1)(63t^5-42t^4+t^3-2t^2-8t-4)} & \frac{4(14t^5+3t^4-4t^3-1)}{(t+1)(63t^5-42t^4+t^3-2t^2-8t-4)}. \end{bmatrix}.$$

The OMP inverse $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$ of A is defined by

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger = BUCAA^\dagger = B(CAB)^{-1}CAA^\dagger,$$

which gives

$$A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = \begin{bmatrix} \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} & \frac{t(15t^4+t^3+6t^2-2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-15t^4+13t^3-8t^2+4t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^2(-20t^3+6t^2+3t-1)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} \\ \frac{t(-2t^4+3t^3+8t^2-3t-2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-66t^5+13t^4-25t^3+2t^2+12t+4}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(3t^4+t^3+16t^2-8t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^2(67t^3-9t^2-18t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-2t^4+3t^3+8t^2-3t-2)}{63t^5-42t^4+t^3-2t^2-8t-4} \\ \frac{t(-16t^4+13t^3-6t^2+3t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(39t^4-7t^3+8t^2-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-24t^5+7t^4+15t^3-6t^2+4t+4}{-63t^5+42t^4-t^3+2t^2+8t+4} & \frac{t^2(-31t^3+13t^2+2t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-16t^4+13t^3-6t^2+3t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} \\ \frac{t(2t^4-3t^3+t^2-5t+1)}{-63t^5+42t^4-t^3+2t^2+8t+4} & -\frac{t(3t^4+29t^3+6t^2+18t+4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(3t^4+t^3-2t^2+10t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^3(4t^2+33t-1)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(2t^4-3t^3+t^2-5t+1)}{-63t^5+42t^4-t^3+2t^2+8t+4} \\ \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} & \frac{t(15t^4+t^3+6t^2-2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-15t^4+13t^3-8t^2+4t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^2(-20t^3+6t^2+3t-1)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} \end{bmatrix}.$$

The MPO inverse $A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ of A is equal to

$$A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^\dagger AB(CAB)^{-1}C,$$

which produces $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. Further calculation gives $A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

Example 5.6.2. Choose the same matrices A, B, C as in Example 5.6.1 in the case $t = -1$. In this case $\operatorname{rank}(CAB) = 2 < 3 = \operatorname{rank}(C)$. According to Urquhart formula, the OMP, MPO and MPOMP inverses are not defined. But, in this case one can verify $\operatorname{rank}(CAB) = 2 = \operatorname{rank}(B)$. According to Proposition 5.5.1, it arises that $B(CAB)^\dagger C \in A\{2\}_{\mathcal{R}(B),*}$ is defined, where

$$(CAB)^\dagger = \begin{bmatrix} -\frac{27}{58} & \frac{16}{29} & \frac{21}{116} \\ \frac{71}{116} & -\frac{41}{58} & -\frac{13}{58} \\ -\frac{27}{58} & \frac{16}{29} & \frac{21}{116} \end{bmatrix}.$$

So, it is possible to consider the following generalization of the OMP inverse

$$X = B(CAB)^\dagger CAA^\dagger = \begin{bmatrix} \frac{3}{29} & \frac{21}{116} & -\frac{41}{116} & -\frac{25}{116} & \frac{3}{29} \\ -\frac{14}{29} & -\frac{69}{116} & \frac{85}{116} & \frac{49}{116} & -\frac{14}{29} \\ \frac{3}{29} & \frac{21}{116} & -\frac{41}{116} & -\frac{25}{116} & \frac{3}{29} \\ \frac{3}{29} & \frac{21}{116} & -\frac{41}{116} & -\frac{25}{116} & \frac{3}{29} \\ \frac{3}{29} & \frac{21}{116} & -\frac{41}{116} & -\frac{25}{116} & \frac{3}{29} \end{bmatrix} \in A^{(2)}_{\mathcal{R}(B),*}AA^\dagger.$$

Similarly, a generalization of the MPO inverse $A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ is defined by $A^\dagger AB(CAB)^\dagger C$, while generalization of the MPOMP inverse $A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$ is defined by $A^\dagger AB(CAB)^\dagger CAA^\dagger$.

Most general form of Φ_1-OMP inverses is $\Phi_1 AA^\dagger$. In order to find $(CAB)^{(1)}$, observe the general solution to the matrix equation $BUCAB = B$, which is given by

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & \frac{3u_{2,1}}{2} - \frac{13}{8} & 1 - 2u_{2,1} \\ u_{3,1} & \frac{1}{2}\left(3u_{1,1} - 2u_{1,2} + 3u_{3,1} + 5\right) & -2u_{1,1} - u_{1,3} - 2u_{3,1} - \frac{3}{2} \end{bmatrix} \subseteq (CAB)\{1\}.$$

Then

$$AA^\dagger = BUCAA^\dagger \subseteq A\{2 : \mathcal{R}(B), *\} =$$

$$\begin{bmatrix} -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) & u_{1,1}+u_{2,1}+u_{3,1}+\frac{1}{2} & \frac{1}{8}\left(12u_{1,1}+12u_{2,1}+12u_{3,1}+1\right) & \frac{1}{8}\left(-4u_{1,1}-4u_{2,1}-4u_{3,1}-3\right) & -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) \\ \frac{1}{8}\left(-12u_{1,1}-36u_{2,1}-12u_{3,1}+7\right) & u_{1,1}+3u_{2,1}+u_{3,1}-\frac{3}{2} & \frac{1}{8}\left(12u_{1,1}+36u_{2,1}+12u_{3,1}-5\right) & \frac{1}{8}\left(-4u_{1,1}-12u_{2,1}-4u_{3,1}+7\right) & \frac{1}{8}\left(-12u_{1,1}-36u_{2,1}-12u_{3,1}+7\right) \\ -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) & u_{1,1}+u_{2,1}+u_{3,1}+\frac{1}{2} & \frac{1}{8}\left(12u_{1,1}+12u_{2,1}+12u_{3,1}+1\right) & \frac{1}{8}\left(-4u_{1,1}-4u_{2,1}-4u_{3,1}-3\right) & -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) \\ -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) & u_{1,1}+u_{2,1}+u_{3,1}+\frac{1}{2} & \frac{1}{8}\left(12u_{1,1}+12u_{2,1}+12u_{3,1}+1\right) & \frac{1}{8}\left(-4u_{1,1}-4u_{2,1}-4u_{3,1}-3\right) & -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) \\ -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) & u_{1,1}+u_{2,1}+u_{3,1}+\frac{1}{2} & \frac{1}{8}\left(12u_{1,1}+12u_{2,1}+12u_{3,1}+1\right) & \frac{1}{8}\left(-4u_{1,1}-4u_{2,1}-4u_{3,1}-3\right) & -\frac{3}{8}\left(4u_{1,1}+4u_{2,1}+4u_{3,1}+1\right) \end{bmatrix}.$$

It is important to mention that $U \subseteq (CAB)\{1\}$ becomes $(CAB)^\dagger$ under particular settings

$$\left\{u_{1,1} \to -\frac{27}{58}, u_{1,2} \to \frac{16}{29}, u_{1,3} \to \frac{21}{116}, u_{2,1} \to \frac{71}{116}, u_{3,1} \to -\frac{27}{58}\right\}.$$

Example 5.6.3. Let us observe the two-variable matrix

$$A = \begin{bmatrix} \frac{1}{b} & a & 0 \\ 0 & \frac{1}{b} & a \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{b} & a & 0 \\ \frac{1}{b} & a & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathrm{rank}(CAB) = 1 = \mathrm{rank}(C) < 2 = \mathrm{rank}(B)$, corresponding OMP, MPO and MPOMP inverses are not defined. But, on the basis of $\mathrm{rank}(CAB) = 1 = \mathrm{rank}(C)$ and according to Algorithm 5.5.2, it arises that $CABUC = C$ is solvable and the set $A^\dagger ABUC \subseteq A\{2 : *, \mathcal{N}(C)\}$ is defined. The solution $U \subseteq (CAB)\{1\}$ to $CABUC = C$ is

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & -\frac{2bu_{3,1}a^2+2bu_{2,1}a+u_{2,1}a+2bu_{2,2}\,a+u_{2,2}a-b^2+u_{1,1}+u_{1,2}}{2a^2b} & u_{3,3} \end{bmatrix}. \tag{5.16}$$

Then

$$A^\dagger ABUC \subseteq A\{2 : *, \mathcal{N}(CAA^\dagger)\} =$$

$$\begin{bmatrix} \frac{a^2\left(b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}\right)b^2+2\left(u_{1,1}+u_{1,2}+a\left(u_{2,1}+u_{2,2}\right)\right)}{2\left(a^4b^5+a^2b^3+b\right)} & \frac{a\left(a^2\left(b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}\right)b^2+2\left(u_{1,1}+u_{1,2}+a\left(u_{2,1}+u_{2,2}\right)\right)\right)}{2\left(a^4b^4+a^2b^2+1\right)} & 0 \\ -\frac{-a^4\left(b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}\right)b^4-b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}}{2b^2\left(b^4a^5+b^2a^3+a\right)} & -\frac{-a^4\left(b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}\right)b^4-b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}}{2\left(a^4b^5+a^2b^3+b\right)} & 0 \\ -\frac{2b^2u_{2,1}a^3+2b^2u_{2,2}a^3+u_{2,1}a+u_{2,2}a-b^2+\left(2a^2b^2+1\right)u_{1,1}+\left(2a^2b^2+1\right)u_{1,2}}{2\left(a^4b^5+a^2b^3+b\right)} & -\frac{a\left(2b^2u_{2,1}a^3+2b^2u_{2,2}a^3+u_{2,1}a+u_{2,2}a-b^2+\left(2a^2b^2+1\right)u_{1,1}+\left(2a^2b^2+1\right)u_{1,2}\right)}{2\left(a^4b^4+a^2b^2+1\right)} & 0 \end{bmatrix}. \tag{5.17}$$

On the other hand, Algorithm 5.5.1 detects the situation $\mho_{CAB,CA}$ and solves $CABUCA = CA$. Its solution U is given as in (5.16). The output is given by $X := BUCAA^\dagger \subseteq A\{2\}_{*,\mathcal{N}(CAA^\dagger)}$, which is equal to

$$B(CAB)\{1\}CAA^\dagger = \begin{bmatrix} \frac{u_{1,1}+u_{1,2}+a\left(u_{2,1}+u_{2,2}\right)}{b} & a\left(u_{1,1}+u_{1,2}+a\left(u_{2,1}+u_{2,2}\right)\right) & 0 \\ -\frac{-b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}}{2ab^2} & -\frac{-b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}}{2b} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathrm{rank}(CAB) = 1 = \mathrm{rank}(CA)$, according to Algorithm 5.5.1, it arises that $CABUC = CA$ is solvable with the solution U defined in (5.16). Then $X := BUCAA^\dagger \subseteq A\{2\}_{*,\mathcal{N}(CAA^\dagger)}$ is equal to

$$BUCAA^{\dagger} = \begin{bmatrix} \frac{u_{1,1}+u_{1,2}+a(u_{2,1}+u_{2,2})}{b} & a\left(u_{1,1}+u_{1,2}+a\left(u_{2,1}+u_{2,2}\right)\right) & 0 \\ -\frac{-b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}}{2ab^2} & -\frac{-b^2+u_{1,1}+u_{1,2}+au_{2,1}+au_{2,2}}{2b} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, Algorithm 5.5.3 detects the case $\mho_{CAB,CA}$, solves $CABUCA = CA$ and generates U defined in (5.16). The output of Algorithm 5.5.3 is given by $X := A^{\dagger}ABUCAA^{\dagger} \subseteq A\{2 : *, \mathcal{N}(CAA^{\dagger})\}$, which is equal to the output (7.12).

Example 5.6.4. Consider

$$A = \begin{bmatrix} 0.0982798 & -0.529283 & 0.328422 & 0.557778 & -0.032553 \\ 0.131677 & -0.8022 & 0.454439 & 0.131041 & -0.856559 \\ 0.0127279 & 1.36991 & -0.897724 & 1.30206 & -0.153377 \\ 0.171995 & 0.277825 & -0.360948 & 1.34927 & -0.0922307 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.603559 & 0.272327 & 0.464822 \\ 0.645163 & 0.87561 & 0.849444 \\ -0.861373 & 0.370456 & 0.283891 \\ -0.83329 & 0.958362 & 0.0983674 \\ 0.351969 & 0.0731028 & 0.151112 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.442855 & 0.199782 & -0.199495 & 0.566626 \\ -0.21934 & -0.0512642 & 0.252624 & -0.214142 \\ -0.153748 & -0.0999656 & -0.0294663 & -0.239401 \end{bmatrix}.$$

Since $\operatorname{rank}(CAB) = 2 = \operatorname{rank}(C) < 3 = \operatorname{rank}(B)$, according to Urquhart formula, $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ does not exist, so that the OMP, MPO and MPOMP inverses are not defined and according to Proposition 5.5.1,

$$B(CAB)^{\dagger}C = \begin{bmatrix} 0.145627 & 0.103134 & 0.055161 & 0.238537 \\ -0.278847 & 0.02053 & 0.597607 & -0.152721 \\ 0.244146 & 0.14312 & -0.00360018 & 0.358374 \\ 0.256173 & 0.200789 & 0.159503 & 0.44662 \\ -0.124237 & -0.0332741 & 0.129421 & -0.127202 \end{bmatrix}$$

is defined and satisfies $B(CAB)^{\dagger}C \in A^{(2)}_{*,\mathcal{N}(C)}$. On the basis of $\mho_{CAB,C}$ and Theorem 5.5.1, it arises that

$$\begin{aligned} X &:= B(CAB)^{\dagger}CAA^{\dagger} \\ &= \begin{bmatrix} 0.145627 & 0.103134 & 0.055161 & 0.238537 \\ -0.278847 & 0.02053 & 0.597607 & -0.152721 \\ 0.244146 & 0.14312 & -0.00360018 & 0.358374 \\ 0.256173 & 0.200789 & 0.159503 & 0.44662 \\ -0.124237 & -0.0332741 & 0.129421 & -0.127202 \end{bmatrix} \end{aligned}$$

satisfies $X \in A_{\Gamma_{\dagger}}$.

Moreover, it is possible to find the set of outer inverses with the prescribed range equal to $B(CAB)\{1\}CAA^{\dagger} = A^{(2)}_{*,\mathcal{N}(C)}AA^{\dagger}$. The solution $U \subseteq (CAB)\{1\}$ to $CABUCA = CA$ is

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & -1.25208u_{1,1} + 1.2974u_{1,2} + 8.659 * 10^{-17}u_{1,3} + 0.9651u_{2,1} - 0.0814033 & -1.9508u_{1,1} - 3.0997 * 10^{-17}u_{1,2} + 1.2974u_{1,3} + 1.5036u_{2,1} - 2.5572 \\ u_{3,1} & 3.08017u_{1,1} - 3.19164u_{1,2} + 6.961 * 10^{-17}u_{1,3} + 1.472 * 10^{-16}u_{2,1} + 0.96507u_{3,1} + 3.6979 & 4.79896u_{1,1} - 5.4502 * 10^{-17}u_{1,2} - 3.19164u_{1,3} + 6.2776 * 10^{-17}u_{2,1} + 1.5036u_{3,1} + 1.84287 \end{bmatrix}. \tag{5.18}$$

Then

$BUCAA^\dagger =$

$$\left[\begin{matrix} -0.7678u_{1,1}+0.3802u_{1,2}+0.2666u_{1,3}-0.3968 & -0.3464u_{1,1}+0.0889u_{1,2}+0.1733u_{1,3}-0.1030 \\ -0.4118u_{1,1}+0.2040u_{1,2}+0.143u_{1,3}-0.569786 & -0.1858u_{1,1}+0.0477u_{1,2}+0.0930u_{1,3}-0.09003 \\ -0.5699u_{1,1}+0.2822u_{1,2}+0.1978u_{1,3}-0.1584 & -0.2571u_{1,1}+0.0660u_{1,2}+0.1287u_{1,3}-0.0099 \\ 0.0426u_{1,1}-0.0211u_{1,2}-0.0148u_{1,3}+0.2862 & 0.0192u_{1,1}-0.0049u_{1,2}-0.0096u_{1,3}+0.212218 \\ -0.0157u_{1,1}+0.0078u_{1,2}+0.005u_{1,3}-0.1353 & -0.0071u_{1,1}+0.00182u_{1,2}+0.0035u_{1,3}-0.0375 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.3459u_{1,1}-0.438u_{1,2}+0.0511u_{1,3}+0.424 & -0.9824u_{1,1}+0.3713u_{1,2}+0.415u_{1,3}-0.4017 \\ 0.1855u_{1,1}-0.2349u_{1,2}+0.027u_{1,3}+0.795 & -0.5269u_{1,1}+0.1991u_{1,2}+0.2226u_{1,3}-0.496 \\ 0.2567u_{1,1}-0.325u_{1,2}+0.0379u_{1,3}+0.27009 & -0.72915u_{1,1}+0.276u_{1,2}+0.308u_{1,3}-0.117 \\ -0.0192u_{1,1}+0.02428u_{1,2}-0.0028u_{1,3}+0.1391 & 0.0545u_{1,1}-0.0206u_{1,2}-0.023u_{1,3}+0.482 \\ 0.0071u_{1,1}-0.00896u_{1,2}+0.001u_{1,3}+0.13697 & -0.02u_{1,1}+0.00765u_{1,2}+0.0085u_{1,3}-0.1403 \end{matrix}\right]$$

$$\subseteq B(CAB)\{1\}CAA^\dagger \subseteq A\{2:*,\mathcal{N}(CAA^\dagger)\}.$$

On the other hand, Algorithm 5.5.2 requires solution of $CABUC = C$, which general form is the same as in (5.18). Then the MPO subclass of the form $A^\dagger ABUC \subseteq A\{2:*,\mathcal{N}(C)\}$ is defined by

$$\left[\begin{matrix} 0.1186u_{1,1}-0.0587u_{1,2}-0.04u_{1,3}+0.1347 & 0.0535u_{1,1}-0.0137u_{1,2}-0.02677u_{1,3}+0.019 \\ -0.1074u_{1,1}+0.053u_{1,2}+0.0373u_{1,3}-0.3872 & -0.048u_{1,1}+0.0124u_{1,2}+0.0242u_{1,3}-0.0481 \\ -0.21006u_{1,1}+0.104u_{1,2}+0.0729u_{1,3}+0.0573 & -0.095u_{1,1}+0.0243u_{1,2}+0.0474u_{1,3}+0.0397 \\ -0.0348u_{1,1}+0.0172u_{1,2}+0.01208u_{1,3}+0.2399 & -0.0157u_{1,1}+0.004u_{1,2}+0.00786u_{1,3}+0.2016 \\ 0.0144635u_{1,1}-0.00716u_{1,2}-0.005u_{1,3}-0.117 & 0.0065u_{1,1}-0.00167u_{1,2}-0.0033u_{1,3}-0.0333 \end{matrix}\right.$$

$$\left.\begin{matrix} -0.05u_{1,1}+0.06766u_{1,2}-0.00789u_{1,3}-0.19495 & 0.1518u_{1,1}-0.05736u_{1,2}-0.064u_{1,3}+0.114 \\ 0.048363u_{1,1}-0.061u_{1,2}+0.0071u_{1,3}+0.5828 & -0.137u_{1,1}+0.0519u_{1,2}+0.058u_{1,3}-0.3189 \\ 0.0946u_{1,1}-0.1198u_{1,2}+0.014u_{1,3}+0.01888 & -0.269u_{1,1}+0.1016u_{1,2}+0.114u_{1,3}+0.0926 \\ 0.0157u_{1,1}-0.01985u_{1,2}+0.00232u_{1,3}+0.193 & -0.04u_{1,1}+0.0168u_{1,2}+0.0188u_{1,3}+0.4371 \\ -0.0065u_{1,1}+0.008u_{1,2}-0.00096u_{1,3}+0.1159 & 0.0185u_{1,1}-0.007u_{1,2}-0.0078u_{1,3}-0.1227 \end{matrix}\right].$$

Finally, Algorithm 5.5.3 requires solution of $CABUCA = CA$, which general form is the same as in (5.18). Then $A^\dagger ABUCAA^\dagger$ produces the subset of Φ_1-MPOMP (identical with the Φ_1-MPO) class.

5.7 Extensions of the generalized CEP inverse

There are different generalizations of the CEP inverse in literature. In [4, 40], the extension of the CEP inverse was given for rectangular matrices, in [97, 104, 106] for bounded linear Hilbert space operators, in [31, 42, 187] for elements of rings and in [128] for tensors. One recent generalization of the CEP inverse by means of the outer inverse was presented in Subsection 5.1.1 for rectangular matrices.

For $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $A \in \mathbb{C}^{m\times n}_{\mathcal{R}(B),\mathcal{N}(C)}$, the generalized CEP (or GCEP) inverse of A is expressed by

$$A^{②}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\left(AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)^\dagger.$$

It was shown that $A^{②}_{\mathcal{R}(B),\mathcal{N}(C)}$ presents the unique solution to the matrix system [113]

$$XAX = X, \qquad AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})^\dagger = AX$$

$$\text{and} \qquad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})^\dagger A = XA.$$

The dual generalized CEP (or *GCEP) inverse of A is given by

$$A_{②}^{\mathcal{R}(B),\mathcal{N}(C)} := \left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A\right)^\dagger A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)},$$

which is the unique solution to

$$XAX = X, \qquad \left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A\right)^\dagger A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A = XA$$

and

$$A\left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A\right)^\dagger A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = AX.$$

Notice that GCEP (or *GCEP) inverse coincides with the CEP (dual core) inverse when $m = n$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{\mathrm{D}}$. The gMP inverse and the dual gMP (or *gMP) inverse of $A \in \mathbb{C}^{n\times n}$

are defined in [147] by the expressions $A^{\diamond} := \left(A^{\textcircled{\dagger}}A\right)^{\dagger} A^{\textcircled{\dagger}}$ and $A_{\diamond} := A_{\textcircled{\dagger}}\left(AA_{\textcircled{\dagger}}\right)^{\dagger}$, respectively. Clearly, the gMP or *gMP inverse are appearances of $A^{\textcircled{2}}_{\mathcal{R}(B),\mathcal{N}(C)}$ and $A^{\mathcal{R}(B),\mathcal{N}(C)}_{\textcircled{2}}$, respectively, in the case $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{\textcircled{\dagger}}$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A_{\textcircled{\dagger}}$.

The main aim of this section is to present the most general forms of the GCEP, *GCEP, CEP and *CEP inverses. Motivated by the very famous Urquhart expression of outer inverses, the extensions of the notions for the GCEP and *GCEP inverses will be introduced in our research.

Remark that, under additional assumptions, the expression $\Phi_1 := B(CAB)^{(1)}C$ is equal to outer inverses $A^{(2)}_{\mathcal{R}(B),*}$, $A^{(2)}_{*,\mathcal{N}(C)}$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. The Urquhart representations (5.9) inspired us to propose and consider Φ_1-extensions of GCEP and *GCEP inverses which are based on the replacement of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ by the more general expression $\Phi_1 = B(CAB)^{(1)}C$. As a consequence, Φ_1-extensions of GCEP inverse (Φ_1-GCEP inverses) as well as Φ_1-extensions of *GCEP inverse (Φ_1-*GCEP inverses) will be introduced using outer inverses with predefined range $A\{2\}_{\mathcal{R}(B),*}$ and/or null space $A\{2\}_{*,\mathcal{N}(C)}$. Moreover, the existence of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ is restricted by the constraint $\mho_{CAB,B,C}$, which is not satisfied for all choices B and C, while Φ_1 exists in all cases.

Further, it will be useful to replace the term $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ in the definitions of the GCEP and *GCEP by the more general expressions $\Phi_2 := B(CAB)^{(2)}C$ and $\Phi \in \mathbb{C}^{n\times m}$. In this way, we obtain generalizations of the GCEP and *GCEP inverses, termed as Φ_2-GCEP and Φ_2-*GCEP inverses or Φ-GCEP and Φ-*GCEP inverses. The results of these section are proved in [115].

5.7.1 Φ_1-GCEP inverse

In this subsection, we suppose that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $(CAB)^{(1)} \in (CAB)\{1\}$ is a fixed but arbitrary. In the expression of the GCEP inverse $A^{\textcircled{2}}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\left(AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)^{\dagger}$ changing the outer inverse $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ with $\Phi_1 = B(CAB)^{(1)}C$, we present Φ_1-GCEP inverse with the more general form $\Phi_1(A\Phi_1)^{\dagger}$.

Definition 5.7.1. *The Φ_1-GCEP inverse of A is defined by*

$$A^{\textcircled{2},(1)}_{B,C} := B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger}. \tag{5.19}$$

Theorem 5.7.1. *The Φ_1-GCEP inverse of A, defined by* (5.19)*, satisfies the following properties:*
(1) $A^{\textcircled{2},(1)}_{B,C} \in A\{2,3\}$;
(2) $\mho_{CAB,B} \Longleftrightarrow A^{\textcircled{2},(1)}_{B,C} = A^{(2)}_{\mathcal{R}(B),*}\left(AA^{(2)}_{\mathcal{R}(B),*}\right)^{\dagger} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}\left(\left(AA^{(2)}_{\mathcal{R}(B),*}\right)^{*}\right)}$;
(3) $\mho_{CAB,C} \Longleftrightarrow A^{\textcircled{2},(1)}_{B,C} = A^{(2)}_{*,\mathcal{N}(C)}\left(AA^{(2)}_{*,\mathcal{N}(C)}\right)^{\dagger} = A^{(2)}_{*,\mathcal{N}\left(\left(AA^{(2)}_{*,\mathcal{N}(C)}\right)^{*}\right)}$;
(4) $\mho_{CAB,B,C} \Longleftrightarrow A^{\textcircled{2},(1)}_{B,C} = A^{\textcircled{2}}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}\left(\left(AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)^{*}\right)}$;
(5) $\mho_{CAB,B,C,A} \Longleftrightarrow A^{\textcircled{2},(1)}_{B,C} = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}\left(AA^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)^{\dagger} = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}\left(\left(AA^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)^{*}\right)}$.

Proof. The statement (1) follows from basic properties of the Moore-Penrose inverse. Parts (2)-(5) follow on the basis of Proposition 5.5.1 and [113, Lemma 2.1(iii)]. □

The Φ_1-GCEP inverse can be characterized in the following way.

Theorem 5.7.2. *For $X \in \mathbb{C}^{n\times m}$, the next claims are mutually equivalent:*
(i) $X = \Phi_1(A\Phi_1)^{\dagger}$;
(ii) $XAX = X$, $AX = A\Phi_1(A\Phi_1)^{\dagger}$ *and* $XA = \Phi_1(A\Phi_1)^{\dagger}A$;
(iii) $AX = A\Phi_1(A\Phi_1)^{\dagger}$ *and* $X = \Phi_1(A\Phi_1)^{\dagger}AX$;
(iv) $A^{\dagger}AX = A^{\dagger}A\Phi_1(A\Phi_1)^{\dagger}$ *and* $X = \Phi_1(A\Phi_1)^{\dagger}AX$;
(v) $A^{*}AX = A^{*}A\Phi_1(A\Phi_1)^{\dagger}$ *and* $X = \Phi_1(A\Phi_1)^{\dagger}AX$;
(vi) $(A\Phi_1)^{\dagger}AX = (A\Phi_1)^{\dagger}$ *and* $\Phi_1(A\Phi_1)^{\dagger}AX = X$;
(vii) $(A\Phi_1)^{*}AX = (A\Phi_1)^{*}$ *and* $\Phi_1(A\Phi_1)^{\dagger}AX = X$;
(viii) $XA = \Phi_1(A\Phi_1)^{\dagger}A$ *and* $XA\Phi_1(A\Phi_1)^{\dagger} = X$;

(ix) $XAA^* = \Phi_1 (A\Phi_1)^\dagger AA^*$ *and* $XA\Phi_1 (A\Phi_1)^\dagger = X$;

(x) $BB^{(1)}X = X$ *and* $B^{(1)}X = B^{(1)}\Phi_1 (A\Phi_1)^\dagger$, *where* $B^{(1)} \in B\{1\}$;

(xi) $XAA^\dagger = X$ *and* $XA = \Phi_1 (A\Phi_1)^\dagger A$.

Proof. (i) $\Rightarrow$ (ii)–(xi): Using (5.19), this part can be proved.

(iii) $\Rightarrow$ (i): By $AX = A\Phi_1 (A\Phi_1)^\dagger$ and $X = \Phi_1 (A\Phi_1)^\dagger AX$, we get

$$X = \Phi_1 (A\Phi_1)^\dagger (AX) = \Phi_1 (A\Phi_1)^\dagger A\Phi_1 (A\Phi_1)^\dagger = \Phi_1 (A\Phi_1)^\dagger .$$

Analogously, the rest can be verified. $\square$

5.7.2 Φ_2-GCEP inverse

We suppose that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $(CAB)^{(2)} \in (CAB)\{2\}$ is a fixed but arbitrary. Replacing $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ with $\Phi_2 = B(CAB)^{(2)}C$ in the definition of the GCEP inverse, we present a new generalization of GCEP inverse with the more general form $\Phi_2(A\Phi_2)^\dagger = B(CAB)^{(2)}C\left(AB(CAB)^{(2)}C\right)^\dagger$.

Definition 5.7.2. *The Φ_2-GCEP inverse of A is defined by*

$$A^{\textcircled{2},(2)}_{B,C} := B(CAB)^{(2)}C\left(AB(CAB)^{(2)}C\right)^\dagger = \Phi_2(A\Phi_2)^\dagger. \tag{5.20}$$

Corollary 5.7.1. *The Φ_2-GCEP inverse of A, defined by* (5.20), *satisfies:*
(1) *There exist suitable Hermitian idempotents P and Q, such that*

$$\begin{aligned} A^{\textcircled{2},(2)}_{B,C} &= B(PCABQ)^\dagger C\left(AB(PCABQ)^\dagger C\right)^\dagger \\ &= BQ(PCABQ)^\dagger PC\left(ABQ(PCABQ)^\dagger PC\right)^\dagger; \end{aligned} \tag{5.21}$$

(2) $A^{\textcircled{2},(2)}_{B,C} = A^{(2)}_{\mathcal{R}(BQ),*}\left(AA^{(2)}_{\mathcal{R}(BQ),*}\right)^\dagger \Longleftrightarrow \mho_{PCABQ,BQ}$;

(3) $A^{\textcircled{2},(2)}_{B,C} = A^{(2)}_{*,\mathcal{N}(PC)}\left(AA^{(2)}_{*,\mathcal{N}(C)}\right)^\dagger \Longleftrightarrow \mho_{PCAB,PC}$;

(4) $A^{\textcircled{2},(2)}_{B,C} = A^{(2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)}\left(AA^{(2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)}\right)^\dagger = A^{\textcircled{2}}_{\mathcal{R}(BQ),\mathcal{N}(PC)} \Longleftrightarrow \mho_{PCABQ,BQ,PC}$;

(5) $A^{\textcircled{2},(2)}_{B,C} = A^{(1,2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)}\left(AA^{(1,2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)}\right)^\dagger \Longleftrightarrow \mho_{PCABQ,BQ,PC,A}$.

Proof. According to [8], $Y \in (CAB)\{2\}$ if and only if $Y = (PCABQ)^\dagger$, where P and Q are suitable Hermitian idempotents. Therefore, $(CAB)^{(2)} = (PCABQ)^\dagger = Q(PCABQ)^\dagger = (PCABQ)^\dagger P$ which implies (5.21). The rest is a consequence of Theorem 5.7.1. $\square$

As in Theorem 5.7.2, we can obtain characterizations for the Φ_2-GCEP inverse changing Φ_1 with Φ_2. Some additional equivalent conditions for Φ_2-GCEP inverse are presented in Theorem 5.7.3.

Theorem 5.7.3. *For $X \in \mathbb{C}^{n\times m}$, the next claims are mutually equivalent:*

(i) $X = A^{\textcircled{2},(2)}_{B,C}$;

(ii) $\Phi_2 AX = X$ *and* $AX = A\Phi_2 (A\Phi_2)^\dagger$;

(iii) $\Phi_2 AX = X$ *and* $A^\dagger AX = A^\dagger A\Phi_2 (A\Phi_2)^\dagger$;

(iv) $\Phi_2 AX = X$ *and* $A^* AX = A^* A\Phi_2 (A\Phi_2)^\dagger$.

Proof. (i) $\Rightarrow$ (ii): Since $X = \Phi_2 (A\Phi_2)^\dagger$, we have $\Phi_2 AX = \Phi_2 A\Phi_2 (A\Phi_2)^\dagger = X$ and $AX = A\Phi_2 (A\Phi_2)^\dagger$.

(ii) $\Rightarrow$ (i): From $\Phi_2 AX = X$ and $AX = A\Phi_2 (A\Phi_2)^\dagger$, we observe that $X = \Phi_2 A\Phi_2 (A\Phi_2)^\dagger = \Phi_2 (A\Phi_2)^\dagger$.

The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are evident by properties of the Moore-Penrose inverse. $\square$

5.7.3 Φ-GCEP inverse

Consider $A \in \mathbb{C}^{m\times n}$ and $\Phi \in \mathbb{C}^{n\times s}$. Using $\Phi \in \mathbb{C}^{n\times s}$ instead of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ in the GCEP inverse, we give the most general Φ-GCEP inverse of A.

Definition 5.7.3. *Consider $A \in \mathbb{C}^{m\times n}$ and $\Phi \in \mathbb{C}^{n\times s}$. The Φ-GCEP inverse of A is defined by*

$$A^{\textcircled{2},\Phi} := \Phi(A\Phi)^{\dagger}. \tag{5.22}$$

Let $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$. Three important particular cases for Φ-GCEP inverses of A are the choices $\Phi = B(CAB)^{(1)}C$, $\Phi = B(CAB)^{(2)}C$ and $\Phi = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, which lead to already presented extensions $A^{\textcircled{2},\Phi} = A^{\textcircled{2},(1)}_{B,C}$, $A^{\textcircled{2},\Phi} = A^{\textcircled{2},(2)}_{B,C}$, and $A^{\textcircled{2},\Phi} = A^{\textcircled{2}}_{B,C}$, respectively.

The Φ-GCEP inverse can be characterized as in Theorem 5.7.2 stating Φ instead of Φ_1 in the statements (i)-(ix) and (xi).

Representations, characterizations and projectors determined by the Φ-GCEP inverse are investigated in Lemma 5.7.1. The set of right inverses of A is denoted by $A^{-1}_{\{R\}} = \{X|\ AX = I\}$.

Lemma 5.7.1. *The following statements hold:*

(i) $AA^{\textcircled{2},\Phi}$ *is the orthogonal projector onto* $\mathcal{R}(A\Phi)$;

(ii) $A^{\textcircled{2},\Phi}A$ *is a projector onto* $\mathcal{R}(\Phi(A\Phi)^*)$ *along* $\mathcal{N}((A\Phi)^*A)$;

(iii) $A^{\textcircled{2},\Phi} = A^{(2,3)}_{\mathcal{R}(\Phi(A\Phi)^*),\mathcal{N}((A\Phi)^*)}$;

(iv) $A^{\textcircled{2},\Phi} = A^{(2,3)}_{\mathcal{R}(\Phi),\mathcal{N}((A\Phi)^*)} \Longleftrightarrow \mho_{A\Phi,\Phi}$;

(v) $A^{\textcircled{2},\Phi} \in A\{1,2,3\} \Longleftrightarrow \mho_{A\Phi,A}$;

(vi) $A^{\textcircled{2},\Phi} \in A\{2,3\}$;

(vii) $A^{\textcircled{2},\Phi} \in A\{2,3\}_s \Longleftrightarrow \Phi \in \mathbb{C}^{n\times s}_s$ and $\mho_{A\Phi,\Phi}$;

(viii) $A^{\textcircled{2},\Phi} \in A^{-1}_{\{R\}} \Longleftrightarrow \Phi \in \mathbb{C}^{n\times m}_m$ and $\mho_{A\Phi,\Phi}$.

Proof. (i) From (5.22), it follows $AA^{\textcircled{2},\Phi} = A\Phi(A\Phi)^{\dagger}$ is the orthogonal projector onto $\mathcal{R}(A\Phi)$.

(ii) Evidently, $A^{\textcircled{2},\Phi}A$ is a projector by the equality $A^{\textcircled{2},\Phi}AA^{\textcircled{2},\Phi} = A^{\textcircled{2},\Phi}$. Because

$$\mathcal{R}(A^{\textcircled{2},\Phi}A) = \mathcal{R}(\Phi(A\Phi)^{\dagger}A) \subseteq \mathcal{R}(\Phi(A\Phi)^{\dagger}) = \mathcal{R}(\Phi(A\Phi)^*)$$

and

$$\mathcal{R}(\Phi(A\Phi)^*) = \mathcal{R}(\Phi(A\Phi)^{\dagger}A\Phi(A\Phi)^{\dagger}) \subseteq \mathcal{R}(\Phi(A\Phi)^{\dagger}A) = \mathcal{R}(A^{\textcircled{2},\Phi}A),$$

we observe that $\mathcal{R}(A^{\textcircled{2},\Phi}A) = \mathcal{R}(\Phi(A\Phi)^*)$. Similarly, we check

$$\mathcal{N}(A^{\textcircled{2},\Phi}A) = \mathcal{N}(\Phi(A\Phi)^{\dagger}A) = \mathcal{N}((A\Phi)^{\dagger}A) = \mathcal{N}((A\Phi)^*A).$$

(iii) It follows by parts (i) and (ii).

(v) This statement follows from $A\Phi(A\Phi)^{\dagger}A = A \Longleftrightarrow \operatorname{rank}(A\Phi) = \operatorname{rank}(A)$ [8]. □

As a consequence of Lemma 5.7.1, we describe projectors involving the Φ_1-GCEP inverse or Φ_2-GCEP inverse.

Corollary 5.7.2. *The following statements hold:*

(i) $AA^{\textcircled{2},(1)}_{B,C}$ *is the orthogonal projector onto* $\mathcal{R}(AB(CAB)^{(1)}C)$;

(ii) $A^{\textcircled{2},(1)}_{B,C}A$ *is a projector onto* $\mathcal{R}(B(CAB)^{(1)}C(AB(CAB)^{(1)}C)^*)$ *along* $\mathcal{N}((AB(CAB)^{(1)}C)^*A)$;

(iii) $A^{\textcircled{2},(1)}_{B,C} = A^{(2)}_{\mathcal{R}(B(CAB)^{(1)}C(AB(CAB)^{(1)}C)^*),\mathcal{N}((AB(CAB)^{(1)}C)^*)}$;

(iv) $AA^{\textcircled{2},(2)}_{B,C}$ *is the orthogonal projector onto* $\mathcal{R}(AB(CAB)^{(2)})$;

(v) $A^{\textcircled{2},(2)}_{B,C}A$ *is a projector onto* $\mathcal{R}(B(CAB)^{(2)}C((CAB)^{(2)}C)^*)$ *along* $\mathcal{N}((AB(CAB)^{(2)})^*A)$;

(vi) $A^{\textcircled{2},(2)}_{B,C} = A^{(2)}_{\mathcal{R}(B(CAB)^{(2)}C((CAB)^{(2)}C)^*),\mathcal{N}((AB(CAB)^{(2)})^*)}$.

Proof. This result follows taking $\Phi = \Phi_1 = B(CAB)^{(1)}C$ or $\Phi = \Phi_2 = B(CAB)^{(2)}C$ in Lemma 5.7.1. □

We also can introduce the Φ-GCEP inverse based on a geometrical point of view.

Theorem 5.7.4. *The matrix* $X := A^{\textcircled{2},\Phi}$ *is the unique solution to the restricted equation*

$$AX = P_{\mathcal{R}(A\Phi)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(\Phi(A\Phi)^*). \tag{5.23}$$

Proof. Lemma 5.7.1 implies that $A^{\textcircled{2},\Phi}$ satisfies (5.23).

For two matrices $X, X_1 \in \mathbb{C}^{n\times m}$ which satisfy (5.23) and $H = X - X_1$, firstly note that $AH = AX - AX_1 = P_{\mathcal{R}(A\Phi)} - P_{\mathcal{R}(A\Phi)} = 0$.
Therefore, $\mathcal{R}(H) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(\Phi(A\Phi)^*A)$, that is, $\Phi(A\Phi)^*AU = 0$. By $\mathcal{R}(X) \subseteq \mathcal{R}(\Phi(A\Phi)^*)$ and $\mathcal{R}(X_1) \subseteq \mathcal{R}(\Phi(A\Phi)^*)$, we deduce that

$$\mathcal{R}(H) \subseteq \mathcal{R}(\Phi(A\Phi)^*) = \mathcal{R}(\Phi\Phi^*A^*) = \mathcal{R}(\Phi\Phi^*A^*A) = \mathcal{R}(\Phi(A\Phi)^*A).$$

Hence, $U = \Phi(A\Phi)^*AV$, for some $V \in \mathbb{C}^{n\times m}$, and

$$\begin{aligned} H &= \Phi(A\Phi)^\dagger A\left(\Phi(A\Phi)^*AV\right) = \Phi(A\Phi)^\dagger AH \\ &= \Phi(A\Phi)^\dagger \left((A\Phi)^\dagger\right)^* (A\Phi)^*AH \\ &= \Phi(A\Phi)^\dagger \left((A\Phi)^\dagger\right)^* (A\Phi)^\dagger A\left(\Phi(A\Phi)^*AH\right) \\ &= 0. \end{aligned}$$

The conclusion is $X = X_1$ and so $A^{\textcircled{2},\Phi}$ is the unique solution to (5.23). $\square$

Theorem 5.7.4 yields the following properties for Φ_1-GCEP and Φ_2-GCEP inverses.

Corollary 5.7.3. *The matrix*

(i) $X := A^{\textcircled{2},(1)}_{B,C}$ *is the unique solution to the restricted equation*

$$AX = P_{\mathcal{R}\left(AB(CAB)^{(1)}C\right)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}\left(B(CAB)^{(1)}C(AB(CAB)^{(1)}C)^*\right);$$

(ii) $X := A^{\textcircled{2},(2)}_{B,C}$ *is the unique solution to the restricted equation*

$$AX = P_{\mathcal{R}(AB(CAB)^{(2)})} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}\left(B(CAB)^{(2)}C((CAB)^{(2)}C)^*\right).$$

5.8 Extensions of dual generalized CEP inverses

In this section, the definitions of the Φ_1-*GCEP and Φ_2-*GCEP inverses are introduced to extend the notation of the *GCEP inverse. The results of this section are given without proofs because they can be verified similarly as the corresponding results in Section 5.7.

5.8.1 Φ_1-*GCEP inverse

Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $(CAB)^{(1)} \in (CAB)\{1\}$ be a fixed but arbitrary. Firstly, we consider the Φ_1-*GCEP inverse as an extension of the *GCEP inverse obtained replacing $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ with Φ_1 in the definition for *GCEP inverse.

Definition 5.8.1. *The* Φ_1*-*GCEP inverse of* A *is defined by*

$$A^{B,C}_{\textcircled{2},(1)} := \left(B(CAB)^{(1)}CA\right)^\dagger B(CAB)^{(1)}C = (\Phi_1A)^\dagger\,\Phi_1. \tag{5.24}$$

Theorem 5.8.1. *The* Φ_1*-*GCEP inverse* $A^{B,C}_{\textcircled{2},(1)}$ *satisfies*

(1) $A^{B,C}_{\textcircled{2},(1)} \in A\{2\}$;

(2) $\mho_{CAB,B} \Longleftrightarrow A^{B,C}_{\textcircled{2},(1)} = \left(A^{(2)}_{\mathcal{R}(B),*}A\right)^\dagger A^{(2)}_{\mathcal{R}(B),*} = A^{(2)}_{\mathcal{R}\left(\left(A^{(2)}_{\mathcal{R}(B),*}A\right)^*\right),*}$;

(3) $\mho_{CAB,C} \Longleftrightarrow A^{B,C}_{\textcircled{2},(1)} = \left(A^{(2)}_{*,\mathcal{N}(C)}A\right)^\dagger A^{(2)}_{*,\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}\left(\left(A^{(2)}_{*,\mathcal{N}(C)}A\right)^*\right),\mathcal{N}(C)}$;

(4) $\mho_{CAB,B,C} \Longleftrightarrow A^{B,C}_{\textcircled{2},(1)} = A^{\mathcal{R}(B),\mathcal{N}(C)}_{\textcircled{2}} = A^{(2)}_{\mathcal{R}\left(\left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A\right)^*\right),\mathcal{N}(C)}$;

(5) $\mho_{CAB,B,C,A} \Longleftrightarrow A^{B,C}_{\textcircled{2},(1)} = \left(A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}A\right)^\dagger A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(1,2)}_{\mathcal{R}\left(\left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}A\right)^*\right),\mathcal{N}(C)}$.

Theorem 5.8.2 includes characterizations of Φ_1-g*-CEP inverse and can be shown as Theorem 5.7.2.

Theorem 5.8.2. *For $X \in \mathbb{C}^{n\times m}$, the next claims are mutually equivalent:*

(i) X *is represented as in* (5.24)*;*

(ii) $XAX = X$, $AX = A(\Phi_1 A)^{\dagger}\Phi_1$ *and* $XA = (\Phi_1 A)^{\dagger}\Phi_1 A$*;*

(iii) $AX = A(\Phi_1 A)^{\dagger}\Phi_1$ *and* $X = (\Phi_1 A)^{\dagger}\Phi_1 AX$*;*

(iv) $A^*AX = A^*A(\Phi_1 A)^{\dagger}\Phi_1$ *and* $X = (\Phi_1 A)^{\dagger}\Phi_1 AX$*;*

(v) $XA(\Phi_1 A)^{\dagger} = (\Phi_1 A)^{\dagger}$ *and* $(\Phi_1 A)^{\dagger}\Phi_1 AX = X$*;*

(vi) $XA(\Phi_1 A)^{*} = (\Phi_1 A)^{*}$ *and* $(\Phi_1 A)^{\dagger}\Phi_1 AX = X$*;*

(vii) $XA = (\Phi_1 A)^{\dagger}\Phi_1 A$ *and* $XA(\Phi_1 A)^{\dagger}\Phi_1 = X$*;*

(viii) $XAA^{\dagger} = (\Phi_1 A)^{\dagger}\Phi_1 AA^{\dagger}$ *and* $XA(\Phi_1 A)^{\dagger}\Phi_1 = X$*;*

(ix) $XAA^{*} = (\Phi_1 A)^{\dagger}\Phi_1 AA^{*}$ *and* $XA(\Phi_1 A)^{\dagger}\Phi_1 = X$*;*

(x) $X = XC^{(1)}C$ *and* $XC^{(1)} = (\Phi_1 A)^{\dagger}\Phi_1 C^{(1)}$*, where* $C^{(1)} \in C\{1\}$*;*

(xi) $A^{\dagger}AX = X$ *and* $AX = A(\Phi_1 A)^{\dagger}\Phi_1$.

5.8.2 Φ_2-*GCEP inverse

In this subsection, we suppose that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $(CAB)^{(2)} \in (CAB)\{2\}$ be a fixed but arbitrary. The Φ_2-*GCEP inverse will be defined changing $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ with Φ_2 in the definition for *GCEP inverse.

Definition 5.8.2. *The Φ_2-*GCEP inverse of A is defined by*

$$A^{B,C}_{②,(2)} := \left(B(CAB)^{(2)}CA\right)^{\dagger} B(CAB)^{(2)}C = (\Phi_2 A)^{\dagger}\Phi_2. \tag{5.25}$$

Corollary 5.8.1. *The Φ_2-*GCEP inverse of A satisfies:*
(1) *There exist suitable Hermitian idempotents P and Q such that*

$$\begin{aligned} A^{B,C}_{②,(2)} &= \left(B(PCABQ)^{\dagger}CA\right)^{\dagger} B(PCABQ)^{\dagger}C \\ &= \left(BQ(PCABQ)^{\dagger}PCA\right)^{\dagger} BQ(PCABQ)^{\dagger}PC; \end{aligned} \tag{5.26}$$

(2) $A^{B,C}_{②,(2)} = \left(A^{(2)}_{\mathcal{R}(BQ),*}A\right)^{\dagger} A^{(2)}_{\mathcal{R}(BQ),*} \iff \mho_{PCABQ,BQ}$;

(3) $A^{B,C}_{②,(2)} = \left(A^{(2)}_{*,\mathcal{N}(PC)}A\right)^{\dagger} A^{(2)}_{*,\mathcal{N}(PC)} \iff \mho_{PCABQ,PC}$;

(4) $A^{B,C}_{②,(2)} = \left(A^{(2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)}A\right)^{\dagger} A^{(2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)} = A^{\mathcal{R}(BQ),\mathcal{N}(PC)}_{②} \iff \mho_{PCABQ,BQ,PC}$;

(5) $A^{B,C}_{②,(2)} = \left(A^{(1,2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)}A\right)^{\dagger} A^{(1,2)}_{\mathcal{R}(BQ),\mathcal{N}(PC)} \iff \mho_{PCABQ,BQ,PC,A}$.

The Φ_2-*GCEP inverse can be characterized as in Theorem 5.8.2 (with Φ_2 instead of Φ_1) and Theorem 5.8.3.

Theorem 5.8.3. *For $X \in \mathbb{C}^{n\times m}$, the following claims are mutually equivalent:*

(i) $X = A^{B,C}_{②,(2)}$*;*

(ii) $XA\Phi_2 = X$ *and* $XA = (\Phi_2 A)^{\dagger}\Phi_2 A$*;*

(iii) $XA\Phi_2 = X$ *and* $XAA^{\dagger} = (\Phi_2 A)^{\dagger}\Phi_2 AA^{\dagger}$*;*

(iv) $XA\Phi_2 = X$ *and* $XAA^{*} = (\Phi_2 A)^{\dagger}\Phi_2 AA^{*}$.

5.8.3 Φ-*GCEP inverse

Suppose that $A \in \mathbb{C}^{m\times n}$, $\Phi \in \mathbb{C}^{s\times m}$. The most general form of *GCEP inverse $(\Phi A)^{\dagger}\Phi$ can be obtained replacing $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ in the definition of *GCEP inverse with Φ.

Definition 5.8.3. *The Φ-*GCEP inverse of A is defined by*

$$A_{②,\Phi} := (\Phi A)^{\dagger}\Phi. \tag{5.27}$$

Let $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$. Three important particular cases for Φ-*GCEP inverses of A are the choices $\Phi = B(CAB)^{(1)}C$, $\Phi = B(CAB)^{(2)}C$ and $\Phi = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, which lead to already considered extensions $A_{\textcircled{2},\Phi} = A^{B,C}_{\textcircled{2},(1)}$, $A_{\textcircled{2},\Phi} = A^{B,C}_{\textcircled{2},(2)}$, and $A_{\textcircled{2},\Phi} = A^{B,C}_{\textcircled{2}}$, respectively.

The Φ-*GCEP inverse is characterized by statements (i)-(ix) and (xi) of Theorem 5.8.2 taking Φ instead of Φ_2.

Representations, characterizations and projectors related to the Φ-*GCEP inverse analogously as in Lemma 5.7.1. The set of left inverses of A is denoted by $A^{-1}_{\{L\}} = \{X|\ XA = I\}$.

Lemma 5.8.1. *The following statements hold:*

(i) $AA_{\textcircled{2},\Phi}$ *is a projector onto* $\mathcal{R}(A(\Phi A)^*)$ *along* $\mathcal{N}((\Phi A)^*\Phi)$;

(ii) $A_{\textcircled{2},\Phi}A$ *is the orthogonal projector onto* $\mathcal{R}((\Phi A)^*)$;

(iii) $A_{\textcircled{2},\Phi} = A^{(2,4)}_{\mathcal{R}((\Phi A)^*),\mathcal{N}((\Phi A)^*\Phi)}$;

(iv) $A_{\textcircled{2},\Phi} = A^{(2,4)}_{\mathcal{R}((\Phi A)^*),\mathcal{N}(\Phi)} \Longleftrightarrow \mho_{\Phi A,\Phi}$;

(v) $A_{\textcircled{2},\Phi} \in A\{1,2,4\} \Longleftrightarrow \mho_{\Phi A,A}$;

(vi) $A_{\textcircled{2},\Phi} \in A\{2,4\}$;

(vii) $A_{\textcircled{2},\Phi} \in A\{2,4\}_s \Longleftrightarrow \Phi \in \mathbb{C}^{s\times m}_s \bigwedge \mho_{\Phi A,\Phi}$;

(viii) $A_{\textcircled{2},\Phi} \in A^{-1}_{\{L\}} \Longleftrightarrow \Phi \in \mathbb{C}^{n\times m}_n \bigwedge \mho_{\Phi A,\Phi}$.

Consequently, we obtain the following results related with projectors determined by the Φ_1-*GCEP and Φ_2-*GCEP inverses.

Corollary 5.8.2. *The following statements hold:*

(i) $AA^{B,C}_{\textcircled{2},(1)}$ *is a projector onto* $\mathcal{R}\left(A(B(CAB)^{(1)}CA)^*\right)$ *along* $\mathcal{N}\left((B(CAB)^{(1)}CA)^*B(CAB)^{(1)}C\right)$;

(ii) $A^{B,C}_{\textcircled{2},(1)}A$ *is the orthogonal projector onto* $\mathcal{R}\left((B(CAB)^{(1)}CA)^*\right)$;

(iii) $A^{B,C}_{\textcircled{2},(1)} = A^{(2)}_{\mathcal{R}\left((B(CAB)^{(1)}CA)^*\right),\mathcal{N}\left((B(CAB)^{(1)}CA)^*B(CAB)^{(1)}C\right)}$;

(iv) $AA^{B,C}_{\textcircled{2},(2)}$ *is a projector onto* $\mathcal{R}(A((CAB)^{(2)}CA)^*)$ *along* $\mathcal{N}\left(\left(B(CAB)^{(2)}\right)^*B(CAB)^{(2)}C\right)$;

(v) $A^{B,C}_{\textcircled{2},(2)}A$ *is the orthogonal projector onto* $\mathcal{R}\left(((CAB)^{(2)}CA)^*\right)$;

(vi) $A^{B,C}_{\textcircled{2},(2)} = A^{(2)}_{\mathcal{R}\left(((CAB)^{(2)}CA)^*\right),\mathcal{N}\left((B(CAB)^{(2)})^*B(CAB)^{(2)}C\right)}$.

From a geometrical approach, the Φ-*GCEP inverse can be characterized as follows.

Theorem 5.8.4. *The unique solution to the restricted equation*

$$AX = P_{\mathcal{R}(A(\Phi A)^*),\mathcal{N}((\Phi A)^*\Phi)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$

is the matrix $X := A_{\textcircled{2},\Phi}$.

We can characterize the Φ_1-*GCEP and Φ_2-*GCEP inverses using Theorem 5.8.4.

Corollary 5.8.3. *The matrix*

(i) $X := A^{B,C}_{\textcircled{2},(1)}$ *presents unique solution of restricted equation*

$$AX = P_{\mathcal{R}\left(A(B(CAB)^{(1)}CA)^*\right),\mathcal{N}\left((B(CAB)^{(1)}CA)^*B(CAB)^{(1)}C\right)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*);$$

(ii) $X := A^{B,C}_{\textcircled{2},(2)}$ *presents unique solution of restricted equation*

$$AX = P_{\mathcal{R}\left(A((CAB)^{(2)}CA)^*\right),\mathcal{N}\left((B(CAB)^{(2)})^*B(CAB)^{(2)}C\right)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*).$$

5.9 Algorithms and examples

Algorithm 5.5.1 is aimed to calculating Φ_1-GCEP inverse and Φ_1-*GCEP generalized inverses. The underlying fact is that the matrix equation LME_B: $BUCAB = B$ is solvable under the conditions $\mho_{CAB,B}$, while LME_C: $CABUC = C$ is solvable in the case $\mho_{CAB,CA}$. If these conditions are not satisfied, then the equation LME_{CAB}: $CABUCAB = CAB$ is always solvable. If U is a general solution obtained by a computer algebra system (for example using *Mathematica* standard function `Solve [173]`), then the general solution to all three matrix equations LME_B,LME_C and LME_{CAB} satisfies $U \subseteq (CAB)\{1\}$, so that $\Phi_1 := BUC \subseteq B(CAB)\{1\}C$, and $\Phi_1 (A\Phi_1)^\dagger$ or $(\Phi_1 A)^\dagger \Phi_1$ is the output. In the case when a general solution requires a great CPU time, then a representative particular solution U to LME_B or LME_C or LME_{CAB} can be generated using *Mathematica* function `FindInstance` [173]. In all three cases, it follows $U \in (CAB)\{1\}$, so that $\Phi_1 := BUC \in B(CAB)\{1\}C$ and again $\Phi_1 (A\Phi_1)^\dagger$ or $(\Phi_1 A)^\dagger \Phi_1$ is desired output. In this way, Algorithm 5.5.1 represents a continuation and extension of the main idea used in the computational procedures developed in [132, 133].

Algorithm 5.9.1 Computing Φ_1-GCEP inverse and Φ_1-*GCEP inverse.

Input: $A \in \mathbb{C}(\mathbf{x})^{m\times n}$, $B \in \mathbb{C}(\mathbf{x})^{n\times p}$, $C \in \mathbb{C}(\mathbf{x})^{q\times m}$.
if $\mho_{CAB}^{B,C}$ **then**
  Solve $CABUCAB = CAB$.
  Compute $\Phi_1 := BUC$.
else if $\mho_{CAB,B}$ **then**
  Solve $BUCAB = B$.
  Compute $\Phi_1 := BUC$.
else if $\mho_{CAB,C}$ **then**
  Solve $CABUC = C$.
  Compute $\Phi_1 := BUC$.
end if
Compute $X1 := \Phi_1 (A\Phi_1)^\dagger$.
Compute $X2 := (\Phi_1 A)^\dagger \Phi_1$.
Output: $X1$ and $X2$.

Algorithm 5.9.2 is aimed to calculating Φ_2-GCEP inverse and Φ_2-*GCEP generalized inverses. Main idea is to solve the matrix equation $VCABV = V$ symbolically, which gives $V \subseteq (CAB)\{2\}$. Then $\Phi_2 := BVC \subseteq B(CAB)\{2\}C$ and $\Phi_2 (A\Phi_2)^\dagger$ or $(\Phi_2 A)^\dagger \Phi_2$ is desired output. If the general solution is too complicated, then V can be extracted as particular solution given by *Mathematica* function `FindInstance`. Then $V \in (CAB)\{2\}$, so that $\Phi_2 := BVC \subseteq B(CAB)\{2\}C$ and $\Phi_2 (A\Phi_2)^\dagger$ or $(\Phi_2 A)^\dagger \Phi_2$ is the output.

Algorithm 5.9.2 Computing Φ_2-GCEP inverse and Φ_2-*GCEP inverse.

Input: $A \in \mathbb{C}(\mathbf{x})^{m\times n}$, $B \in \mathbb{C}(\mathbf{x})^{n\times p}$, $C \in \mathbb{C}(\mathbf{x})^{q\times m}$.
Solve $VCABV = V$.
Compute $\Phi_2 := BVC$.
Compute $X1 := \Phi_2 (A\Phi_2)^\dagger$.
Compute $X2 := (\Phi_2 A)^\dagger \Phi_2$.
Output: $X1$ and $X2$.

Example 5.9.1. *Consider*

$$A = \begin{bmatrix} 4 & 3 & 3 & 3 & 4 \\ 3 & 2 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 2 & 3 \\ 4 & 3 & 3 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 3 & 3 \\ 3 & 5 & 3 \\ 3 & 3 & 7 \\ 3 & 3 & 3 \\ 7 & 3 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & 9 & 9 & 9 & 10 \\ 9 & 8 & 9 & 9 & 9 \\ 9 & 9 & 10 & 9 & 9 \end{bmatrix}.$$

Solution to the matrix equation $CABUC = C$ is

$$U = \begin{bmatrix} \frac{313}{1486} & -\frac{951}{2972} & \frac{135}{1486} \\ -\frac{2811}{5944} & \frac{1475}{1486} & -\frac{693}{1486} \\ \frac{459}{2972} & -\frac{1413}{2972} & \frac{221}{743} \end{bmatrix} \in A\{1\}$$

and

$$\Phi_1 = BUC = \begin{bmatrix} \frac{467}{2972} & \frac{1941}{5944} & -\frac{1383}{5944} & -\frac{2151}{5944} & \frac{467}{2972} \\ -\frac{15}{2972} & -\frac{7801}{5944} & \frac{579}{5944} & \frac{7515}{5944} & -\frac{15}{2972} \\ -\frac{741}{2972} & \frac{4557}{5944} & \frac{2449}{5944} & -\frac{3231}{5944} & -\frac{741}{2972} \\ -\frac{57}{2972} & -\frac{1707}{5944} & \frac{417}{5944} & \frac{1809}{5944} & -\frac{57}{2972} \\ \frac{467}{2972} & \frac{1941}{5944} & -\frac{1383}{5944} & -\frac{2151}{5944} & \frac{467}{2972} \end{bmatrix}.$$

Now,

$$A^{\textcircled{2},(1)}_{B,C} = B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger}$$
$$= \begin{bmatrix} \frac{8303}{59750} & \frac{429}{1195} & -\frac{7998}{29875} & -\frac{9273}{29875} & \frac{8303}{59750} \\ \frac{3441}{59750} & -\frac{1702}{1195} & \frac{6519}{29875} & \frac{32469}{29875} & \frac{3441}{59750} \\ -\frac{3309}{11950} & \frac{195}{239} & \frac{2144}{5975} & -\frac{2781}{5975} & -\frac{3309}{11950} \\ -\frac{51}{11950} & -\frac{75}{239} & \frac{591}{5975} & \frac{1566}{5975} & -\frac{51}{11950} \\ \frac{8303}{59750} & \frac{429}{1195} & -\frac{7998}{29875} & -\frac{9273}{29875} & \frac{8303}{59750} \end{bmatrix}$$

and

$$A_{\textcircled{2},(1)}^{B,C} = \left(B(CAB)^{(1)}CA\right)^{\dagger} B(CAB)^{(1)}C = (\Phi_1 A)^{\dagger}\,\Phi_1$$
$$= \begin{bmatrix} \frac{22936}{140221} & -\frac{8439}{280442} & -\frac{64101}{280442} & -\frac{8811}{280442} & \frac{22936}{140221} \\ -\frac{2775}{140221} & -\frac{69493}{140221} & \frac{12342}{140221} & \frac{71154}{140221} & -\frac{2775}{140221} \\ -\frac{33243}{140221} & \frac{12324}{140221} & \frac{58867}{140221} & \frac{11970}{140221} & -\frac{33243}{140221} \\ -\frac{4776}{140221} & \frac{75342}{140221} & \frac{8508}{140221} & -\frac{64449}{140221} & -\frac{4776}{140221} \\ \frac{22936}{140221} & -\frac{8439}{280442} & -\frac{64101}{280442} & -\frac{8811}{280442} & \frac{22936}{140221} \end{bmatrix}.$$

It can be verified that $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C) = 3 < 4 = \operatorname{rank}(A)$, *which guarantees the existence of* $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. *Therefore,* Φ_1*-GCEP inverse* $A^{\textcircled{2},(1)}_{B,C} = B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger}$ *coincides with the GCEP inverse* $A^{\textcircled{2}}_{\mathcal{R}(B),\mathcal{N}(C)}$ *and* Φ_1*-*GCEP inverse* $A^{\textcircled{2},(1)}_{B,C} = (\Phi_1 A)^{\dagger}\Phi_1$ *coincides with the *GCEP inverse* $A^{\mathcal{R}(B),\mathcal{N}(C)}_{\textcircled{2}}$.

On the other hand, one solution to the matrix equation $VCABV = V$ *is*

$$V = \begin{bmatrix} 0 & -\frac{951}{2972} & 0 \\ -\frac{461675}{706593} & 0 & -\frac{693}{1486} \\ -\frac{532413}{2040278} & -\frac{889425}{1020139} & 0 \end{bmatrix} \in (CAB)\{2\}.$$

Then

$$\Phi_2 = BVC = \begin{bmatrix} -\frac{108296223295}{1293536252} & -\frac{24616820658}{323384063} & -\frac{106557786747}{1293536252} & -\frac{104748054669}{1293536252} & -\frac{108296223295}{1293536252} \\ -\frac{363471496661}{3880608756} & -\frac{27823383897}{323384063} & -\frac{118934870991}{1293536252} & -\frac{115918650861}{1293536252} & -\frac{363471496661}{3880608756} \\ -\frac{147497778499}{1293536252} & -\frac{33365782908}{323384063} & -\frac{144409142583}{1293536252} & -\frac{142599410505}{1293536252} & -\frac{147497778499}{1293536252} \\ -\frac{93395312419}{1293536252} & -\frac{21305507130}{323384063} & -\frac{91656875871}{1293536252} & -\frac{89847143793}{1293536252} & -\frac{93395312419}{1293536252} \\ -\frac{108296223295}{1293536252} & -\frac{24616820658}{323384063} & -\frac{106557786747}{1293536252} & -\frac{104748054669}{1293536252} & -\frac{108296223295}{1293536252} \end{bmatrix}.$$

Now, the Φ_2*-GCEP inverse of* A *is equal to*

$$A^{\textcircled{2},(2)}_{B,C} = B(CAB)^{(2)}C\left(AB(CAB)^{(2)}C\right)^{\dagger} = \Phi_2(A\Phi_2)^{\dagger}$$
$$= \begin{bmatrix} \frac{8303}{59750} & \frac{429}{1195} & -\frac{7998}{29875} & -\frac{9273}{29875} & \frac{8303}{59750} \\ \frac{3441}{59750} & -\frac{1702}{1195} & \frac{6519}{29875} & \frac{32469}{29875} & \frac{3441}{59750} \\ -\frac{3309}{11950} & \frac{195}{239} & \frac{2144}{5975} & -\frac{2781}{5975} & -\frac{3309}{11950} \\ -\frac{51}{11950} & -\frac{75}{239} & \frac{591}{5975} & \frac{1566}{5975} & -\frac{51}{11950} \\ \frac{8303}{59750} & \frac{429}{1195} & -\frac{7998}{29875} & -\frac{9273}{29875} & \frac{8303}{59750} \end{bmatrix}$$

and the Φ_2*-*GCEP inverse of* A *is*

$$A_{\textcircled{2},(2)}^{B,C} = \left(B(CAB)^{(2)}CA\right)^{\dagger} B(CAB)^{(2)}C = (\Phi_2 A)^{\dagger}\,\Phi_2$$
$$= \begin{bmatrix} \frac{22936}{140221} & -\frac{8439}{280442} & -\frac{64101}{280442} & -\frac{8811}{280442} & \frac{22936}{140221} \\ -\frac{2775}{140221} & -\frac{69493}{140221} & \frac{12342}{140221} & \frac{71154}{140221} & -\frac{2775}{140221} \\ -\frac{33243}{140221} & \frac{12324}{140221} & \frac{58867}{140221} & \frac{11970}{140221} & -\frac{33243}{140221} \\ -\frac{4776}{140221} & \frac{75342}{140221} & \frac{8508}{140221} & -\frac{64449}{140221} & -\frac{4776}{140221} \\ \frac{22936}{140221} & -\frac{8439}{280442} & -\frac{64101}{280442} & -\frac{8811}{280442} & \frac{22936}{140221} \end{bmatrix}.$$

Example 5.9.2. *Consider A, B, C defined by*

$$A = \begin{bmatrix} 0 & -1 & -1 & -1 & 0 \\ -1 & -2 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -2 & -1 \\ 0 & -1 & -1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -3 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

Ranks of these matrices satisfy $\operatorname{rank}(CAB) = \operatorname{rank}(B) = 2 < \operatorname{rank}(C) = 3 < 4 = \operatorname{rank}(A)$, *which implies that* $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ *does not exist. Therefore, GCEP inverse* $A^{②}_{\mathcal{R}(B),\mathcal{N}(C)}$ *as well as *GCEP inverse* $A_{②}^{\mathcal{R}(B),\mathcal{N}(C)}$ *do not make sense. But, it is possible to define* Φ_1*-GCEP inverse* $A^{②,(1)}_{B,C} = B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger}$ *and* Φ_1*-*GCEP inverse* $A^{②,(1)}_{B,C} = (\Phi_1 A)^{\dagger}\Phi_1$ *of* A.

Due to $\operatorname{rank}(CAB) = \operatorname{rank}(C)$, *the matrix equation* $CABUC = C$ *is consistent. Using the particular solution*

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{13}{8} & 1 \\ 0 & \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \in A\{1\},$$

it is obtained

$$\Phi_1 = BUC = \begin{bmatrix} -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} \\ \frac{7}{8} & -\frac{3}{2} & -\frac{5}{8} & \frac{7}{8} & \frac{7}{8} \\ -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} \end{bmatrix}.$$

Now,

$$A^{②,(1)}_{B,C} = B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \\ -\frac{9}{32} & -\frac{9}{16} & \frac{1}{32} & \frac{21}{32} & -\frac{9}{32} \\ \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \end{bmatrix}$$

and

$$A^{B,C}_{②,(1)} = \left(B(CAB)^{(1)}CA\right)^{\dagger} B(CAB)^{(1)}C = (\Phi_1 A)^{\dagger}\,\Phi_1 = \begin{bmatrix} \frac{2}{17} & -\frac{50}{187} & -\frac{28}{187} & \frac{2}{17} & \frac{2}{17} \\ \frac{1}{17} & -\frac{42}{187} & -\frac{31}{187} & \frac{1}{17} & \frac{1}{17} \\ -\frac{3}{17} & \frac{41}{187} & \frac{8}{187} & -\frac{3}{17} & -\frac{3}{17} \\ -\frac{4}{17} & \frac{49}{187} & \frac{5}{187} & -\frac{4}{17} & -\frac{4}{17} \\ \frac{2}{17} & -\frac{50}{187} & -\frac{28}{187} & \frac{2}{17} & \frac{2}{17} \end{bmatrix}.$$

In order to calculate the Φ_2*-GCEP inverse of* A, *we will explore the solution to the matrix equation* $VCABV = V$ *which is equal to*

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{7}{6} \\ -\frac{5}{3} & 0 & 0 \end{bmatrix} \in (CAB)\{2\}.$$

Then

$$\Phi_2 = BVC = \begin{bmatrix} \frac{9}{2} & \frac{17}{6} & 4 & \frac{17}{6} & \frac{9}{2} \\ \frac{41}{6} & \frac{31}{6} & \frac{26}{3} & \frac{31}{6} & \frac{41}{6} \\ \frac{9}{2} & \frac{17}{6} & 4 & \frac{17}{6} & \frac{9}{2} \\ \frac{9}{2} & \frac{17}{6} & 4 & \frac{17}{6} & \frac{9}{2} \\ \frac{9}{2} & \frac{17}{6} & 4 & \frac{17}{6} & \frac{9}{2} \end{bmatrix}.$$

Now, the Φ_2*-GCEP inverse of* A *is equal to*

$$A^{②,(2)}_{B,C} = B(CAB)^{(2)}C\left(AB(CAB)^{(2)}C\right)^{\dagger} = \Phi_2(A\Phi_2)^{\dagger} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \\ -\frac{9}{32} & -\frac{9}{16} & \frac{1}{32} & \frac{21}{32} & -\frac{9}{32} \\ \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{8} & -\frac{1}{16} & -\frac{5}{16} & \frac{1}{16} \end{bmatrix}$$

*and the Φ_2-*GCEP inverse of A is*

$$A^{B,C}_{\textcircled{2},(2)} = \left(B(CAB)^{(2)}CA\right)^{\dagger} B(CAB)^{(2)}C = (\Phi_2 A)^{\dagger}\,\Phi_2$$
$$= \begin{bmatrix} \frac{22936}{140221} & -\frac{8439}{280442} & -\frac{64101}{280442} & -\frac{8811}{280442} & \frac{22936}{140221} \\ -\frac{2775}{140221} & -\frac{69493}{140221} & \frac{12342}{140221} & \frac{71154}{140221} & -\frac{2775}{140221} \\ -\frac{33243}{140221} & \frac{12324}{140221} & \frac{58867}{140221} & \frac{11970}{140221} & -\frac{33243}{140221} \\ -\frac{4776}{140221} & \frac{75342}{140221} & \frac{8508}{140221} & -\frac{64449}{140221} & -\frac{4776}{140221} \\ \frac{22936}{140221} & -\frac{8439}{280442} & -\frac{64101}{280442} & -\frac{8811}{280442} & \frac{22936}{140221} \end{bmatrix}.$$

Example 5.9.3. *Let*

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 2 & 1 & 1 \\ 6 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Ranks of these matrices satisfy $\operatorname{rank}(CAB) = \operatorname{rank}(A) = 2 < \operatorname{rank}(B) = \operatorname{rank}(C) = 3$, *which implies that* $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ *does not exist. Again, it is possible to define Φ_1-GCEP inverse*

$$A^{\textcircled{2},(1)}_{B,C} = B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger}$$

*and Φ_1-*GCEP inverse $A^{\textcircled{2},(1)}_{B,C} = (\Phi_1 A)^{\dagger}\Phi_1$ of A.*

Due to $\operatorname{rank}(CAB) < \operatorname{rank}(B) = \operatorname{rank}(C)$, *only the equation $CABABUC = CAB$ is consistent with the general solution*

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & \frac{7}{5}\left(28u_{1,1} + 4u_{1,2} - 5\left(u_{2,1}+2\right)\right) & \frac{1}{5}\left(-28u_{1,1} + 28u_{1,3} + 5u_{2,1} + 19\right) \\ u_{3,1} & -\frac{56u_{1,1}}{5} - \frac{8u_{1,2}}{5} - 7u_{3,1} + 3 & \frac{1}{5}\left(8u_{1,1} - 8u_{1,3} + 5u_{3,1} - 4\right) \end{bmatrix} \subseteq A\{1\},$$

which initiates

$$\begin{aligned} \Phi_1 &= BUC \\ &= \left[\begin{matrix} \frac{2}{5}\left(8u_{1,1} - 3u_{1,3} + 5u_{3,1} - 4\right) & \frac{1}{5}\left(13u_{1,1} - 3u_{1,3} + 10u_{3,1} - 4\right) \\ \frac{2}{5}(-4u_{1,1} + 4u_{1,3} + 5u_{2,1} + 15u_{3,1} + 7) & \frac{1}{5}(-4u_{1,1} + 4u_{1,3} + 10u_{2,1} + 30u_{3,1} + 7) \\ \frac{2}{5}\left(8u_{1,1} + 2u_{1,3} + 5u_{3,1} - 4\right) & \frac{2}{5}\left(9u_{1,1} + u_{1,3} + 5u_{3,1} - 2\right) \end{matrix}\right. \\ &\qquad \left.\begin{matrix} \frac{1}{5}(-25u_{1,1} - 3u_{1,2} - 6u_{1,3} - 10u_{3,1} + 7) \\ 4u_{1,1} + \frac{4u_{1,2}}{5} + \frac{8u_{1,3}}{5} - 2u_{2,1} - 6u_{3,1} - \frac{11}{5} \\ \frac{1}{5}\left(-10u_{1,1} + 2u_{1,2} + 4u_{1,3} - 10u_{3,1} + 7\right) \end{matrix}\right]. \end{aligned}$$

Finally,

$$A^{\textcircled{2},(1)}_{B,C} = B(CAB)^{(1)}C\left(AB(CAB)^{(1)}C\right)^{\dagger} = \Phi_1(A\Phi_1)^{\dagger}$$

and

$$A^{B,C}_{\textcircled{2},(1)} = \left(B(CAB)^{(1)}CA\right)^{\dagger} B(CAB)^{(1)}C = (\Phi_1 A)^{\dagger}\,\Phi_1$$

are computed in symbolic form. These forms are not presented because of available space.

In order to calculate the Φ_2-GCEP inverse of A, we will explore the solution to the matrix equation $VCABV = V$ which is equal to

$$V = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -\frac{27}{70} \end{bmatrix} \in (CAB)\{2\}.$$

Then

$$\Phi_2 = BVC = \begin{bmatrix} -\frac{27}{35} & -\frac{27}{70} & -\frac{27}{35} \\ -\frac{81}{35} & -\frac{221}{70} & -\frac{291}{35} \\ -\frac{27}{35} & -\frac{27}{70} & -\frac{27}{35} \end{bmatrix}.$$

Now, the Φ_2-GCEP inverse of A is equal to

$$A^{\textcircled{2},(2)}_{B,C} = B(CAB)^{(2)}C\left(AB(CAB)^{(2)}C\right)^{\dagger} = \Phi_2(A\Phi_2)^{\dagger} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{15} & \frac{2}{15} \\ \frac{2}{3} & -\frac{11}{15} & -\frac{1}{15} \\ -\frac{1}{3} & \frac{7}{15} & \frac{2}{15} \end{bmatrix}$$

*and the Φ_2-*GCEP inverse of A is*

$$A^{B,C}_{\textcircled{2},(2)} = \left(B(CAB)^{(2)}CA\right)^{\dagger} B(CAB)^{(2)}C = (\Phi_2 A)^{\dagger}\,\Phi_2 = \begin{bmatrix} -\frac{20}{203} & \frac{8}{203} & \frac{34}{203} \\ \frac{104}{203} & -\frac{1}{203} & -\frac{55}{203} \\ -\frac{34}{29} & \frac{2}{29} & \frac{23}{29} \end{bmatrix}.$$

5.10 Applicability of Φ-GCEP and Φ-*GCEP inverses

Some applications of the proposed Φ-GCEP and Φ-*GCEP inverses in finding solutions to several linear vector equations are presented in subsequent results. These applications show that general solutions of considered problems involve some of Φ-GCEP and Φ-*GCEP inverses.

An application of Φ-GCEP inverse in solving linear systems is presented in Theorem 5.10.1.

Theorem 5.10.1. *For $A \in \mathbb{C}^{m\times n}$, $\Phi \in \mathbb{C}^{n\times m}$ and $b \in \mathbb{C}^m$, the general solution to*

$$(A\Phi)^* Ax = (A\Phi)^* b \tag{5.28}$$

is expressed as

$$x = A^{\textcircled{2},\Phi} b + \left(I - A^{\textcircled{2},\Phi} A\right) y, \tag{5.29}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Let x have the form (5.29). Since

$$\begin{aligned}(A\Phi)^* Ax &= (A\Phi)^* AA^{\textcircled{2},\Phi} b + (A\Phi)^* A \left(I - A^{\textcircled{2},\Phi} A\right) y \\ &= (A\Phi)^* A\Phi(A\Phi)^\dagger b + (A\Phi)^* A \left(I - \Phi(A\Phi)^\dagger A\right) y \\ &= (A\Phi)^* b,\end{aligned}$$

we deduce that x solves (5.28).

If x is a solution to (5.28), it follows

$$\begin{aligned}A^{\textcircled{2},\Phi} Ax &= \Phi(A\Phi)^\dagger Ax = \Phi(A\Phi)^\dagger \left((A\Phi)^\dagger\right)^* (A\Phi)^* Ax \\ &= \Phi(A\Phi)^\dagger \left((A\Phi)^\dagger\right)^* (A\Phi)^* b \\ &= \Phi(A\Phi)^\dagger b = A^{\textcircled{2},\Phi} b.\end{aligned}$$

Hence,

$$x = A^{\textcircled{2},\Phi} b + x - A^{\textcircled{2},\Phi} Ax = A^{\textcircled{2},\Phi} b + \left(I - A^{\textcircled{2},\Phi} A\right) x,$$

i.e., x is the form (5.29). □

Consequently, Theorem 5.10.1 gives the solvability of the following equations.

Corollary 5.10.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $b \in \mathbb{C}^m$.*

(i) *For a fixed but arbitrary element $(CAB)^{(1)} \in (CAB)\{1\}$, the general solution to*

$$\left(AB(CAB)^{(1)}C\right)^* Ax = \left(AB(CAB)^{(1)}C\right)^* b$$

is expressed as

$$x = A^{\textcircled{2},(1)}_{B,C} b + \left(I - A^{\textcircled{2},(1)}_{B,C} A\right) y,$$

for any $y \in \mathbb{C}^n$.

(ii) *For a fixed but arbitrary element $(CAB)^{(2)} \in (CAB)\{2\}$, the general solution to*

$$\left(AB(CAB)^{(2)}C\right)^* Ax = \left(AB(CAB)^{(2)}C\right)^* b$$

is expressed as

$$x = A^{\textcircled{2},(2)}_{B,C} b + \left(I - A^{\textcircled{2},(1)}_{B,C} A\right) y,$$

for any $y \in \mathbb{C}^n$.

Proof. Choosing $\Phi = \Phi_1 = B(CAB)^{(1)}C$ or $\Phi = \Phi_2 = B(CAB)^{(2)}C$ in Theorem 5.10.1, we verify this result. □

Utilizing the Φ-*GCEP inverse, we establish the general solutions of the next equations as in Theorem 5.10.1 and Corollary 5.10.1.

Theorem 5.10.2. *For $A \in \mathbb{C}^{m\times n}$, $\Phi \in \mathbb{C}^{n\times m}$ and $b \in \mathbb{C}^m$, the general solution to*

$$(\Phi A)^* \Phi Ax = (\Phi A)^* \Phi b$$

is expressed as

$$x = A_{\textcircled{2},\Phi} b + (I - A_{\textcircled{2},\Phi} A)\, y,$$

for any $y \in \mathbb{C}^n$.

Corollary 5.10.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ and $b \in \mathbb{C}^m$.*

(i) *For a fixed but arbitrary element $(CAB)^{(1)} \in (CAB)\{1\}$, the general solution to*

$$\left(B(CAB)^{(1)}CA\right)^* B(CAB)^{(1)}CAx = \left(B(CAB)^{(1)}CA\right)^* B(CAB)^{(1)}Cb$$

is expressed as

$$x = A^{B,C}_{②,(1)}b + \left(I - A^{B,C}_{②,(1)}A\right)y,$$

for arbitrary $y \in \mathbb{C}^n$.

(ii) *For a fixed but arbitrary element $(CAB)^{(2)} \in (CAB)\{2\}$, the general solution to*

$$(B(CAB)^{(2)}CA)^*B(CAB)^{(2)}CAx = (B(CAB)^{(2)}CA)^*B(CAB)^{(2)}Cb$$

is expressed as

$$x = A^{B,C}_{②,(2)}b + \left(I - A^{B,C}_{②,(1)}A\right)y,$$

for arbitrary $y \in \mathbb{C}^n$.

5.11 Summary

In this chapter we investigate extensions of the OMP inverses $A^{(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$, MPO inverses $A^{\dagger,(2)}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ and MPOMP inverses defined by $A^{\dagger,(2),\dagger}_{\mathcal{R}(B),\mathcal{N}(C)} := A^\dagger AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^\dagger$, where $A \in \mathbb{C}^{m\times n}_r$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$ [137]. The extension is based on the replacement of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ involved in the definitions of OMP, MPO and MPOMP classes by the more general expressions $\Phi_1 := B(CAB)^{(1)}C$, whose existence is unconditionally guaranteed. The term Φ_1-*composite outer inverses* will be used to point to obtained generalizations of OMP, MPO and MPOMP inverses. In this way, composite inverses with limited area of definiteness are extended to more general classes which are defined in all cases. Main properties, characterizations and representations of obtained Φ_1-composite outer inverses are investigated. Conditions which enable that Φ_1-composite outer inverses reduce to particular outer inverses with given image and kernel are investigated. Algorithms for numeric and symbolic computation of Φ_1-composite outer inverses are proposed. Corresponding algorithms for calculating the core, core-EP, *core-EP, DMP, MPD, the CMP, MPCEP and *CEPMP inverses can be developed in special cases.

It is important to mention that theoretical analysis and algorithms proposed in [132, 133] investigated the cases $\mho_{CAB,B}$ and $\mho_{CAB,C}$. The extension in there proposed algorithms are investigation of the more general cases $\mho_{CAB,CA}$ and $\mho_{CAB,AB}$.

In addition, Φ_2-composite outer inverses arising from the replacements of the term $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ in composite outer inverses by the expressions $\Phi_2 := B(CAB)^{(2)}C \in A\{2\}$ are investigated.

The initiated further research can include extension of weighted composite outer inverses from [112] on the basis of there proposed principle. Also, extension of obtained results to tensor case will be a challenging topic.

The three kinds of extensions of the GCEP and CEP inverses are presented as the next goal of this investigation based on [113]. These extension are based on the replacement of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ contained in the definition of GCEP inverse by the more general expressions whose existence is unconditionally guaranteed. In particular, utilizing $\Phi_1 := B(CAB)^{(1)}C$, $\Phi_2 := B(CAB)^{(2)}C$ and $\Phi \in \mathbb{C}^{n\times m}$, respectively, instead of $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, the Φ_1-GCEP, Φ_2-GCEP and Φ-GCEP inverses are defined. Under some additional assumptions, these new inverses reduce to the GCEP and CEP inverses and present more general classes which are defined in all cases.

Furthermore, we introduce extensions the *GCEP and *CEP inverses which are marked as Φ_1-*GCEP, Φ_2-*GCEP and Φ-*GCEP inverses.

Corresponding algorithms are derived and illustrative examples are presented. Some applications of the proposed Φ-GCEP and Φ-*GCEP inverses in finding solutions to several linear vector equations are presented.

Numerous options for selecting suitable expressions Φ remained open. Each of future appropriate choices will lead to new classes of generalized inverses, which would be an inspiring topic for future research. Moreover, this chapter calculates inner and outer inversions using an approach based on solving matrix equations. Of course, this is just one of the options for the development of effective computational algorithms. Any of the methods for calculating inner and outer inverses can be used in the development of efficient, or even more efficient algorithms. Each of these approaches can be used in future research. One perspective for future research may be the extension of the obtained results to the tensor case. This would open the possibility of generalizing the results obtained in the paper [141].

Chapter 6

Outer-star and star-outer matrices

Motivated by the DMP inverse and the fact that the conjugate transpose A^* of a given square matrix A and its Moore–Penrose inverse $A^\dagger$ have certain same properties (for example $R(A^\dagger) = R(A^*)$ and $N(A^\dagger) = N(A^*)$), the Drazin-star matrix of A was presented in [95] as a new class of square matrices in the following manner

$$A^{\mathrm{D},*} = A^{\mathrm{D}}AA^*.$$

For $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$, the Drazin-star matrix $A^{\mathrm{D},*}$ of A is the unique solution of the following system of equations

$$X(A^\dagger)^*X = X, \qquad A^kX = A^kA^* \qquad \text{and} \qquad X(A^\dagger)^* = A^{\mathrm{D}}A. \tag{6.1}$$

Notice that the Drazin-star matrix of A (or the Drazin-star inverse of $(A^\dagger)^*$)) is a new class of the outer inverse of $(A^\dagger)^*$ because it is different from each of the Drazin inverse, Moore–Penrose inverse, DMP inverse and MPD inverse of $(A^\dagger)^*$ [95, Example 2.2]. The star-Drazin matrix of A is also defined in [95] as

$$A^{*,\mathrm{D}} = A^*AA^{\mathrm{D}}.$$

Our contribution in this chapter is to proposed two new classes of rectangular matrices in order to solve particular types of matrix equations and generalize the notion of the composite outer inverses, Drazin-star and star-Drazin matrices.

Inspired by similar characterizations of the conjugate transpose of a given rectangular matrix A and its Moore–Penrose inverse $A^\dagger$, we present the outer-star matrix in terms of the outer inverse $A^{(2)}_{T,S}$ and its conjugate transpose A^*. In particular, the outer-star matrix is introduced replacing $A^\dagger$ with A^* in the definition of the OMP inverse of A. Because the Drazin inverse is a particular case of the outer inverse, the outer-star matrix becomes the Drazin-star matrix when $A^{(2)}_{T,S} = A^D$. Hence, we define a new wider class of rectangular matrix.

The second intention is to propose the notion of the star-outer matrix which covers the star-Drazin matrix. Various characterizations of the outer-star and star-outer matrices are developed by algebraic and geometrical point of view. Different representations of outer-star and star-outer matrices such as integral representations and limit representations are proved. The results of this chapter are given in [98].

6.1 Characterizations of outer-star and star-outer matrices

In the beginning of this section, we propose two new classes of rectangular matrices to solve certain types of matrix equations and generalize the notions of Drazin-star and star-Drazin matrices. The defined matrices are proper combinations of outer inverse and the conjugate transpose of a given matrix.

Theorem 6.1.1. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(a) *The system of equations*

$$X(A^\dagger)^*X = X, \qquad AX = AA^{(2)}_{T,S}AA^* \qquad \textit{and} \qquad X(A^\dagger)^* = A^{(2)}_{T,S}A \tag{6.2}$$

is consistent and its unique solution is $X = A^{(2)}_{T,S}AA^*$.

(b) *The system of equations*

$$X(A^\dagger)^*X = X, \qquad XA = A^*AA^{(2)}_{T,S}A \qquad \text{and} \qquad (A^\dagger)^*X = AA^{(2)}_{T,S}$$

is consistent and its unique solution is $X = A^*AA^{(2)}_{T,S}$.

Proof. (a) Set $X = A^{(2)}_{T,S}AA^*$. Then $AX = AA^{(2)}_{T,S}AA^*$,

$$X(A^\dagger)^* = A^{(2)}_{T,S}AA^*(A^\dagger)^* = A^{(2)}_{T,S}AA^\dagger A = A^{(2)}_{T,S}A$$

and

$$X(A^\dagger)^*X = A^{(2)}_{T,S}AX = (A^{(2)}_{T,S}AA^{(2)}_{T,S})AA^* = A^{(2)}_{T,S}AA^* = X$$

imply that $X = A^{(2)}_{T,S}AA^*$ satisfies the system (6.2).

Suppose that (6.2) holds for two matrices X and X_1, that is, $X(A^\dagger)^*X = X$, $X_1(A^\dagger)^*X_1 = X_1$, $AX = AA^{(2)}_{T,S}AA^* = AX_1$ and $X(A^\dagger)^* = A^{(2)}_{T,S}A = X_1(A^\dagger)^*$. Thus, by

$$X = (X(A^\dagger)^*)X = A^{(2)}_{T,S}(AX) = (A^{(2)}_{T,S}A)X_1 = X_1(A^\dagger)^*X_1 = X_1,$$

the system of equations (6.2) has a unique solution.

The part (b) can be checked in an analogy way. $\square$

Definition 6.1.1. *Let* $A \in \mathbb{C}^{m\times n}_{T,S}$.

(a) *The* (T,S)*-outer-star matrix of* A (*or the* (T,S)*-outer-star inverse of* $(A^\dagger)^*$) *is defined as*

$$A^{(2,*)}_{T,S} = A^{(2)}_{T,S}AA^*.$$

(b) *The star-*(T,S)*-outer matrix of* A (*or the star-*(T,S)*-outer inverse of* $(A^\dagger)^*$) *is defined as*

$$A^{(*,2)}_{T,S} = A^*AA^{(2)}_{T,S}.$$

Several particular cases of outer-star and star-outer matrices are listed now.

(i) For $m = n$, $\operatorname{ind}(A) = k$ and $A^{(2)}_{T,S} := A^{\mathrm{D}}$, the system (6.2) reduces to

$$X(A^\dagger)^*X = X, \qquad AX = AA^{\mathrm{D}}AA^* \qquad \text{and} \qquad X(A^\dagger)^* = A^{\mathrm{D}}A. \tag{6.3}$$

Applying Theorem 6.1.1, the matrix $X := A^{\mathrm{D}}AA^* = A^{\mathrm{D},*}$ is the unique solution to (6.3). Thus, the (T,S)-outer-star matrix of A becomes the Drazin-star matrix of A in the case $A^{(2)}_{T,S} := A^{\mathrm{D}}$. By properties of the Drazin inverse, notice that (6.3) implies (6.1). In this case, notice that the star-(T,S)-outer matrix of A coincides with the star-Drazin matrix $X := A^*AA^{\mathrm{D}} = A^{*,D}$, which is, by Theorem 6.1.1, the unique solution to the system

$$X(A^\dagger)^*X = X, \qquad XA = A^*AA^{\mathrm{D}}A \qquad \text{and} \qquad (A^\dagger)^*X = AA^{\mathrm{D}}.$$

(ii) When $m = n$, $\operatorname{ind}(A) = 1$ and $A^{(2)}_{T,S} := A^{\#}$, the system (6.2) is converted into the system

$$X(A^\dagger)^*X = X, \quad AX = AA^*, \quad X(A^\dagger)^* = A^{\#}A,$$

which has the unique solution $X = A^{\#}AA^* = A^{\#,*}$ by Theorem 6.1.1. Thus, the (T,S)-outer-star matrix of A is equal to the group-star matrix of A. Also, the star-(T,S)-outer matrix of A reduces to the star-group matrix $X := A^*AA^{\#} = A^{*,\#}$ in this case and it is uniquely determined solution to the system

$$X(A^\dagger)^*X = X, \qquad XA = A^*A \qquad \text{and} \qquad (A^\dagger)^*X = AA^{\#}.$$

(iii) If $m = n$, $\operatorname{ind}(A) = k$ and $A^{(2)}_{T,S} := A^{\oplus}$, the core–EP-star matrix $A^{\oplus,*} = A^{\oplus}AA^*$ is the unique solution to the matrix system [146]

$$X(A^\dagger)^*X = X, \qquad AX = AA^{\oplus}AA^* \qquad \text{and} \qquad X(A^\dagger)^* = A^{\oplus}A;$$

and the star-core–EP matrix $A^{*,\oplus} = A^*AA^{\oplus}$ is the unique solution to [146]

$$X(A^\dagger)^*X = X, \qquad XA = A^*AA^{\oplus}A \qquad \text{and} \qquad (A^\dagger)^*X = AA^{\oplus}.$$

In general, the outer-star and star-outer matrices are not generalized inverses of a given matrix A, but they are outer inverses of $(A^\dagger)^*$. We now establish ranges and null spaces of outer-star and star-outer matrices, and projections determined by them.

Lemma 6.1.1. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then:*

(i) $(A^\dagger)^* A^{(2,*)}_{T,S}$ *is a projector onto* $\mathcal{R}((A^\dagger)^* A^{(2)}_{T,S})$ *along* $\mathcal{N}(A^{(2)}_{T,S}AA^*)$*;*

(ii) $A^{(2,*)}_{T,S}(A^\dagger)^*$ *is a projector onto T along* $\mathcal{N}(A^{(2)}_{T,S}A)$*;*

(iii) $A^{(2,*)}_{T,S} = [(A^\dagger)^*]^{(2)}_{T,\mathcal{N}(A^{(2)}_{T,S}AA^*)}$*;*

(iv) $(A^\dagger)^* A^{(*,2)}_{T,S}$ *is a projector onto* $\mathcal{R}(AA^{(2)}_{T,S})$ *along S;*

(v) $A^{(*,2)}_{T,S}(A^\dagger)^*$ *is a projector onto* $\mathcal{R}(A^*AA^{(2)}_{T,S})$ *along* $\mathcal{N}(A^{(2)}_{T,S}(A^\dagger)^*)$*;*

(vi) $A^{(*,2)}_{T,S} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A^*AA^{(2)}_{T,S}),S}$.

Proof. (i) Because $A^{(2,*)}_{T,S}(A^\dagger)^* A^{(2,*)}_{T,S} = A^{(2,*)}_{T,S}$, we deduce that $(A^\dagger)^* A^{(2,*)}_{T,S}$ is a projector. The equality $(A^\dagger)^* A^{(2,*)}_{T,S} = (A^\dagger)^* A^{(2)}_{T,S}AA^*$ implies $\mathcal{R}((A^\dagger)^* A^{(2,*)}_{T,S}) \subseteq \mathcal{R}((A^\dagger)^* A^{(2)}_{T,S})$ and

$$\begin{aligned}\mathcal{R}((A^\dagger)^* A^{(2)}_{T,S}) &= \mathcal{R}((A^\dagger)^* A^{(2)}_{T,S}AA^\dagger AA^{(2)}_{T,S})\\ &= \mathcal{R}((A^\dagger)^* A^{(2)}_{T,S}AA^*(A^\dagger)^* A^{(2)}_{T,S})\\ &\subseteq \mathcal{R}((A^\dagger)^* A^{(2,*)}_{T,S}).\end{aligned}$$

Hence, $\mathcal{R}((A^\dagger)^* A^{(2,*)}_{T,S}) = \mathcal{R}((A^\dagger)^* A^{(2)}_{T,S})$. We also observe that $\mathcal{N}(A^{(2)}_{T,S}AA^*) = \mathcal{N}(A^{(2,*)}_{T,S}) = \mathcal{N}(A^{(2,*)}_{T,S}(A^\dagger)^*)$.

(ii) From $A^{(2,*)}_{T,S}(A^\dagger)^* = A^{(2)}_{T,S}A$, we get $\mathcal{R}(A^{(2,*)}_{T,S}(A^\dagger)^*) = \mathcal{R}(A^{(2)}_{T,S}A) = \mathcal{R}(A^{(2)}_{T,S}) = T$ and $\mathcal{N}(A^{(2,*)}_{T,S}(A^\dagger)^*) = \mathcal{N}(A^{(2)}_{T,S}A)$.

(iii) By $\mathcal{R}(A^{(2,*)}_{T,S}) = \mathcal{R}(A^{(2,*)}_{T,S}(A^\dagger)^*) = T$ and $\mathcal{N}(A^{(2,*)}_{T,S}) = \mathcal{N}((A^\dagger)^* A^{(2,*)}_{T,S}) = \mathcal{N}(A^{(2)}_{T,S}AA^*)$, this part follows.

The parts (iv)–(vi) can be proved analogously. □

Lemma 6.1.1 gives the following consequence related to the Drazin-star and star-Drazin inverses.

Corollary 6.1.1. [95] *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. Then:*

(i) $(A^\dagger)^* A^{\mathrm{D},*}$ *is a projector onto* $\mathcal{R}((A^\dagger)^* A^{\mathrm{D}})$ *along* $\mathcal{N}(A^{\mathrm{D}}A^*)$*;*

(ii) $A^{\mathrm{D},*}(A^\dagger)^*$ *is a projector onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}(A^k)$*;*

(iii) $A^{\mathrm{D},*} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^DA^*)}$*;*

(iv) $(A^\dagger)^* A^{*,\mathrm{D}}$ *is a projector onto* $\mathcal{R}(A^k)$ *along* $\mathcal{N}(A^k)$*;*

(v) $A^{*,\mathrm{D}}(A^\dagger)^*$ *is a projector onto* $\mathcal{R}(A^*A^{\mathrm{D}})$ *along* $\mathcal{N}(A^{\mathrm{D}}(A^\dagger)^*)$*;*

(vi) $A^{*,\mathrm{D}} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A^*A^{\mathrm{D}}),\mathcal{N}(A^k)}$.

Theorem 6.1.2 contains a list of equivalent conditions for a rectangular matrix to be an outer-star matrix.

Theorem 6.1.2. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. For $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) *X is the (T,S)-outer-star matrix of A;*

(ii) *X satisfies the equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*A^{(2)}_{T,S}A,$$

$$AX = AA^{(2)}_{T,S}AA^* \quad \text{and} \quad X(A^\dagger)^* = A^{(2)}_{T,S}A;$$

(iii) *X satisfies the equations*

$$A^{(2)}_{T,S}AX = X \quad \text{and} \quad AX = AA^{(2)}_{T,S}AA^*;$$

(iv) *X satisfies the equations*

$$A^{(2)}_{T,S}AXAA^\dagger = X \quad \text{and} \quad AX(A^\dagger)^* = AA^{(2)}_{T,S}A;$$

(v) *X satisfies the equations*

$$A^{(2)}_{T,S}AX = X \quad \text{and} \quad (A^\dagger)^*X = (A^\dagger)^*A^{(2)}_{T,S}AA^*;$$

(vi) X *satisfies the equations*

$$XAA^{\dagger} = X \quad \textit{and} \quad X(A^{\dagger})^{*} = A^{(2)}_{T,S}A;$$

(vii) X *satisfies the equations*

$$X(A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*} = X \quad \textit{and} \quad X(A^{\dagger})^{*}A^{(2)}_{T,S} = A^{(2)}_{T,S};$$

(viii) X *satisfies the equations*

$$XAA^{\dagger} = X \quad \textit{and} \quad X(A^{\dagger})^{*}A^{\dagger} = A^{(2),\dagger}_{T,S};$$

(ix) X *satisfies the equations*

$$XAA^{\dagger} = X \quad \textit{and} \quad XA = A^{(2)}_{T,S}AA^{*}A;$$

(x) X *satisfies equations*

$$X(A^{\dagger})^{*}A^{(2)}_{T,S}AX = X, \quad (A^{\dagger})^{*}A^{(2)}_{T,S}AX = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*}$$
$$\textit{and} \quad X(A^{\dagger})^{*}A^{(2)}_{T,S}A = A^{(2)}_{T,S}A;$$

(xi) X *satisfies the equations*

$$X(A^{\dagger})^{*}A^{(2)}_{T,S}AX = X, \quad (A^{\dagger})^{*}A^{(2)}_{T,S}AX(A^{\dagger})^{*}A^{(2)}_{T,S}A = (A^{\dagger})^{*}A^{(2)}_{T,S}A,$$
$$(A^{\dagger})^{*}A^{(2)}_{T,S}AX = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*} \quad \textit{and} \quad X(A^{\dagger})^{*}A^{(2)}_{T,S}A = A^{(2)}_{T,S}A;$$

(xii) X *satisfies the equations*

$$A^{(2)}_{T,S}AX = X \quad \textit{and} \quad (A^{\dagger})^{*}A^{(2)}_{T,S}AX = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*};$$

(xiii) X *satisfies the equations*

$$X(A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*} = X \quad \textit{and} \quad X(A^{\dagger})^{*}A^{(2)}_{T,S}A = A^{(2)}_{T,S}A.$$

Proof. (i) $\Rightarrow$ (ii): The equality $X = A^{(2)}_{T,S}AA^{*}$ yields

$$(A^{\dagger})^{*}X(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*}(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{\dagger}A = (A^{\dagger})^{*}A^{(2)}_{T,S}A.$$

The rest follows by Theorem 6.1.1.

(ii) $\Rightarrow$ (iii): It is clear by $X = (X(A^{\dagger})^{*})X = A^{(2)}_{T,S}AX$.

(iii) $\Rightarrow$ (iv): From the assumptions $A^{(2)}_{T,S}AX = X$ and $AX = AA^{(2)}_{T,S}AA^{*}$, we obtain

$$A^{(2)}_{T,S}(AX)AA^{\dagger} = A^{(2)}_{T,S}AA^{(2)}_{T,S}A(A^{*}AA^{\dagger}) = A^{(2)}_{T,S}(AA^{(2)}_{T,S}AA^{*}) = A^{(2)}_{T,S}AX = X$$

and

$$(AX)(A^{\dagger})^{*} = AA^{(2)}_{T,S}AA^{*}(A^{\dagger})^{*} = AA^{(2)}_{T,S}AA^{\dagger}A = AA^{(2)}_{T,S}A.$$

(iv) $\Rightarrow$ (i): Using $A^{(2)}_{T,S}AXAA^{\dagger} = X$ and $AX(A^{\dagger})^{*} = AA^{(2)}_{T,S}A$, we get

$$X = A^{(2)}_{T,S}AXAA^{\dagger} = A^{(2)}_{T,S}(AX(A^{\dagger})^{*})A^{*} = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^{*} = A^{(2)}_{T,S}AA^{*}.$$

(i) $\Rightarrow$ (v)–(vi): Applying $X = A^{(2)}_{T,S}AA^{*}$, we have $(A^{\dagger})^{*}X = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*}$ and

$$XAA^{\dagger} = A^{(2)}_{T,S}AA^{*}AA^{\dagger} = A^{(2)}_{T,S}AA^{*} = X.$$

The equalities $A^{(2)}_{T,S}AX = X$ and $X(A^{\dagger})^{*} = A^{(2)}_{T,S}A$ are evident by previous proved equivalences between statements (i)–(iv).

(v) $\Rightarrow$ (i): The assumptions $A^{(2)}_{T,S}AX = X$ and $(A^{\dagger})^{*}X = (A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*}$ give

$$X = A^{(2)}_{T,S}AX = A^{(2)}_{T,S}AA^{*}((A^{\dagger})^{*}X) = A^{(2)}_{T,S}AA^{*}(A^{\dagger})^{*}A^{(2)}_{T,S}AA^{*} = A^{(2)}_{T,S}AA^{*}.$$

We similarly can prove the rest. $\square$

Choosing T and S to be the range and null space of some matrices $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$, respectively, we get characterizations of $(\mathcal{R}(B),\mathcal{N}(C))$-outer-star matrix $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{*}$.

Theorem 6.1.3. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$. If $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ exists (or* $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C)$*), then, for $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) *X is the $(\mathcal{R}(B),\mathcal{N}(C))$-outer-star matrix of A;*

(ii) *X satisfies the equations*

$$CAX = CAA^* \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = X;$$

(iii) *X satisfies the equations*

$$CAX(A^\dagger)^* = CA \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X;$$

(iv) *X satisfies the equations*

$$CAXAA^\dagger = CAA^* \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X;$$

(v) *X satisfies the equations*

$$X(A^\dagger)^*B = B \quad \text{and} \quad X(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X;$$

(vi) *X satisfies the equations*

$$AX(A^\dagger)^*B = AB \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X;$$

(vii) *X satisfies the equations*

$$A^*AX(A^\dagger)^*B = A^*AB \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X.$$

Proof. Firstly, from $R(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}) = \mathcal{R}(B)$, we conclude identities $A^{(2)}_{R(B),N(C)}AB = B$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, for $B^{(1)} \in B\{1\}$. Further, $\mathcal{N}(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}) = \mathcal{N}(C)$ implies $CAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = C$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}C$, for $C^{(1)} \in C\{1\}$.

(i) $\Rightarrow$ (ii): Using $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*$, we get $CAX = (CAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})AA^* = CAA^*$ and

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X.$$

(ii) $\Rightarrow$ (i): Since $CAX = CAA^*$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = X$, then

$$\begin{aligned} X &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}(CAX) \\ &= (A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}C)AA^* = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*. \end{aligned}$$

(i) $\Rightarrow$ (iii): The hypothesis $X = A^{(2)}_{R(B),N(C)}AA^*$ gives $CAX(A^\dagger)^* = CAA^*(A^\dagger)^* = CA$ and

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = XAA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*AA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X.$$

(iii) $\Rightarrow$ (iv): This part is evident.

(iv) $\Rightarrow$ (i): By the assumptions $CAXAA^\dagger = CAA^*$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = X$, we observe that

$$\begin{aligned} X &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AXAA^\dagger = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}(CAXAA^\dagger) \\ &= (A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}C^{(1)}C)AA^* = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*. \end{aligned}$$

(i) $\Rightarrow$ (v): Because $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AB = B$, we see that

$$X(A^\dagger)^*B = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*(A^\dagger)^*B = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AB = B$$

and

$$\begin{aligned} X(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* \\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X. \end{aligned}$$

(v) $\Rightarrow$ (i): Applying $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ and $X(A^\dagger)^*B = B$, we have

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X(A^\dagger)^*(BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}) = X(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}.$$

Therefore, by $X(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X$, $X = (X(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})AA^* = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*$.

(i) $\Rightarrow$ (vi): From $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*$ and $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AB = B$, it follows $AX(A^\dagger)^*B = AB$ and

$$\begin{aligned}&A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*\\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*\\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X.\end{aligned}$$

(vi) $\Rightarrow$ (vii): It is obvious.

(vii) $\Rightarrow$ (i): By $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^* = X$ and $A^*AX(A^\dagger)^*B = A^*AB$, we get

$$\begin{aligned}X &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AX(A^\dagger)^*A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*\\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(A^\dagger)^*(A^*AX(A^\dagger)^*B)B^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*\\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(A^\dagger)^*A^*A(BB^{(1)}A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)})AA^*\\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*\\ &= A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}AA^*.\end{aligned}$$

□

Consequently, we get necessary and sufficient conditions for a square matrix to be the Drazin-star matrix.

Corollary 6.1.2. [95] *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. The following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times n}$ is the Drazin-star matrix of A;*

(ii) *X satisfies equations*
$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*A^{\mathrm{D}}A,$$
$$A^kX = A^kA^* \quad \text{and} \quad X(A^\dagger)^* = A^{\mathrm{D}}A;$$

(iii) *X satisfies equations*
$$A^{\mathrm{D}}AX = X \quad \text{and} \quad A^kX = A^kA^*;$$

(iv) *X satisfies equations*
$$A^{\mathrm{D}}AXAA^\dagger = X \quad \text{and} \quad A^kX(A^\dagger)^* = A^k;$$

(v) *X satisfies equations*
$$A^{\mathrm{D}}AX = X \quad \text{and} \quad AX = AA^{\mathrm{D}}AA^*;$$

(vi) *X satisfies equations*
$$A^{\mathrm{D}}AX = X \quad \text{and} \quad (A^\dagger)^*X = (A^\dagger)^*A^{\mathrm{D}}AA^*;$$

(vii) *X satisfies equations*
$$XAA^\dagger = X \quad \text{and} \quad X(A^\dagger)^* = A^{\mathrm{D}}A;$$

(viii) *X satisfies equations*
$$X(A^\dagger)^*AA^{\mathrm{D}}A^* = X \quad \text{and} \quad X(A^\dagger)^*A^k = A^k;$$

(ix) *X satisfies equations*
$$XAA^\dagger = X \quad \text{and} \quad X(A^\dagger)^*A^\dagger = A^{\mathrm{D},\dagger};$$

(x) *X satisfies equations*
$$XAA^\dagger = X \quad \text{and} \quad XA = A^{\mathrm{D}}AA^*A;$$

(xi) *X satisfies equations*
$$X(A^\dagger)^*A^{\mathrm{D}}AX = X, \quad (A^\dagger)^*A^{\mathrm{D}}AX = (A^\dagger)^*A^{\mathrm{D}}AA^*$$
$$\text{and} \quad X(A^\dagger)^*A^{\mathrm{D}}A = A^{\mathrm{D}}A;$$

(xii) *X satisfies equations*
$$X(A^\dagger)^*A^{\mathrm{D}}AX = X, \quad (A^\dagger)^*A^{\mathrm{D}}AX(A^\dagger)^*A^{\mathrm{D}}A = (A^\dagger)^*A^{\mathrm{D}}A,$$
$$(A^\dagger)^*A^{\mathrm{D}}AX = (A^\dagger)^*A^{\mathrm{D}}AA^* \quad \text{and} \quad X(A^\dagger)^*A^{\mathrm{D}}A = A^{\mathrm{D}}A;$$

(xiii) X *satisfies equations*

$$A^{\mathrm{D}}AX = X \quad \textit{and} \quad (A^{\dagger})^{*}A^{\mathrm{D}}AX = (A^{\dagger})^{*}A^{\mathrm{D}}AA^{*};$$

(xiv) X *satisfies equations*

$$X(A^{\dagger})^{*}AA^{\mathrm{D}}A^{*} = X \quad \textit{and} \quad X(A^{\dagger})^{*}A^{\mathrm{D}}A = A^{\mathrm{D}}A.$$

The following result concerning necessary and sufficient conditions for a rectangular matrix to be the star-outer matrix of a given matrix, can be verified similarly as Theorem 6.1.2.

Theorem 6.1.4. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. For $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) X *is the star-(T,S)-outer matrix of A;*

(ii) X *satisfies the equations*

$$X(A^{\dagger})^{*}X = X, \quad (A^{\dagger})^{*}X(A^{\dagger})^{*} = AA^{(2)}_{T,S}(A^{\dagger})^{*},$$

$$XA = A^{*}AA^{(2)}_{T,S}A \quad \textit{and} \quad (A^{\dagger})^{*}X = AA^{(2)}_{T,S};$$

(iii) X *satisfies the equations*

$$XAA^{(2)}_{T,S} = X \quad \textit{and} \quad XA = A^{*}AA^{(2)}_{T,S}A;$$

(iv) X *satisfies the equations*

$$A^{\dagger}AXAA^{(2)}_{T,S} = X \quad \textit{and} \quad (A^{\dagger})^{*}XA = AA^{(2)}_{T,S}A;$$

(v) X *satisfies the equations*

$$XAA^{(2)}_{T,S} = X \quad \textit{and} \quad X(A^{\dagger})^{*} = A^{*}AA^{(2)}_{T,S}(A^{\dagger})^{*};$$

(vi) X *satisfies the equations*

$$A^{\dagger}AX = X \quad \textit{and} \quad (A^{\dagger})^{*}X = AA^{(2)}_{T,S};$$

(vii) X *satisfies the equations*

$$A^{*}AA^{(2)}_{T,S}(A^{\dagger})^{*}X = X \quad \textit{and} \quad A^{(2)}_{T,S}(A^{\dagger})^{*}X = A^{(2)}_{T,S};$$

(viii) X *satisfies the equations*

$$A^{\dagger}AX = X \quad \textit{and} \quad A^{\dagger}(A^{\dagger})^{*}X = A^{\dagger,(2)}_{T,S};$$

(ix) X *satisfies the equations*

$$A^{\dagger}AX = X \quad \textit{and} \quad AX = AA^{*}AA^{(2)}_{T,S};$$

(x) X *satisfies the equations*

$$XAA^{(2)}_{T,S}(A^{\dagger})^{*}X = X, \quad AA^{(2)}_{T,S}(A^{\dagger})^{*}X = AA^{(2)}_{T,S}$$

$$\textit{and } XAA^{(2)}_{T,S}(A^{\dagger})^{*} = A^{*}AA^{(2)}_{T,S}(A^{\dagger})^{*};$$

(xi) X *satisfies the equations*

$$XAA^{(2)}_{T,S}(A^{\dagger})^{*}X = X, \quad AA^{(2)}_{T,S}(A^{\dagger})^{*}XAA^{(2)}_{T,S}(A^{\dagger})^{*} = AA^{(2)}_{T,S}(A^{\dagger})^{*},$$

$$AA^{(2)}_{T,S}(A^{\dagger})^{*}X = AA^{(2)}_{T,S} \quad \textit{and} \quad XAA^{(2)}_{T,S}(A^{\dagger})^{*} = A^{*}AA^{(2)}_{T,S}(A^{\dagger})^{*};$$

(xii) X *satisfies the equations*

$$XAA^{(2)}_{T,S} = X \quad \textit{and} \quad XAA^{(2)}_{T,S}(A^{\dagger})^{*} = A^{*}AA^{(2)}_{T,S}(A^{\dagger})^{*};$$

(xiii) X *satisfies the equations*

$$A^{*}AA^{(2)}_{T,S}(A^{\dagger})^{*}X = X \quad \textit{and} \quad AA^{(2)}_{T,S}(A^{\dagger})^{*}X = AA^{(2)}_{T,S}.$$

The star-$(\mathcal{R}(B),\mathcal{N}(C))$-outer matrix $A^{*}AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ can be characterized in the next way.

Theorem 6.1.5. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$. If $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ exists, then, for $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) *X is the star-$(\mathcal{R}(B),\mathcal{N}(C))$-outer matrix of A;*

(ii) *X satisfies the equations*

$$XAB = A^*AB \quad \text{and} \quad XAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X;$$

(iii) *X satisfies the equations*

$$(A^\dagger)^*XAB = AB \quad \text{and} \quad A^\dagger AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X;$$

(iv) *X satisfies the equations*

$$A^\dagger AXAB = A^*AB \quad \text{and} \quad A^\dagger AXAA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = X;$$

(v) *X satisfies the equations*

$$C(A^\dagger)^*X = C \quad \text{and} \quad A^*AA^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(A^\dagger)^*X = X;$$

(vi) *X satisfies the equations*

$$C(A^\dagger)^*XA = CA \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(A^\dagger)^*XAA^* = X;$$

(vii) *X satisfies the equations*

$$C(A^\dagger)^*XAA^* = CAA^* \quad \text{and} \quad A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}(A^\dagger)^*XAA^* = X.$$

The next characterizations of the star-Drazin matrix follow as consequences.

Corollary 6.1.3. [95] *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. The following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times n}$ is the star-Drazin matrix of A;*

(ii) *X satisfies equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = A^{\mathrm{D}}A(A^\dagger)^*,$$
$$XA^k = A^*A^k \quad \text{and} \quad (A^\dagger)^*X = A^{\mathrm{D}}A;$$

(iii) *X satisfies equations*

$$XAA^{\mathrm{D}} = X \quad \text{and} \quad XA^k = A^*A^k;$$

(iv) *X satisfies equations*

$$A^\dagger AXAA^{\mathrm{D}} = X \quad \text{and} \quad (A^\dagger)^*XA^k = A^k;$$

(v) *X satisfies equations*

$$XAA^{\mathrm{D}} = X \quad \text{and} \quad XA = A^*AA^{\mathrm{D}}A;$$

(vi) *X satisfies equations*

$$XAA^{\mathrm{D}} = X \quad \text{and} \quad X(A^\dagger)^* = A^*AA^{\mathrm{D}}(A^\dagger)^*;$$

(vii) *X satisfies equations*

$$A^\dagger AX = X \quad \text{and} \quad (A^\dagger)^*X = A^{\mathrm{D}}A;$$

(viii) *X satisfies equations*

$$A^*AA^{\mathrm{D}}(A^\dagger)^*X = X \quad \text{and} \quad A^k(A^\dagger)^*X = A^k;$$

(ix) *X satisfies equations*

$$A^\dagger AX = X \quad \text{and} \quad A^\dagger(A^\dagger)^*X = A^{\dagger,D};$$

(x) *X satisfies equations*

$$A^\dagger AX = X \quad \text{and} \quad AX = AA^*AA^{\mathrm{D}};$$

(xi) *X satisfies equations*

$$XA^{\mathrm{D}}A(A^\dagger)^*X = X, \quad A^{\mathrm{D}}A(A^\dagger)^*X = A^{\mathrm{D}}A$$
$$\text{and} \quad XA^{\mathrm{D}}A(A^\dagger)^* = A^*A^{\mathrm{D}}A(A^\dagger)^*;$$

(xii) X *satisfies equations*

$$XA^{\mathrm{D}}A(A^{\dagger})^{*}X = X, \quad A^{\mathrm{D}}A(A^{\dagger})^{*}XA^{\mathrm{D}}A(A^{\dagger})^{*} = A^{\mathrm{D}}A(A^{\dagger})^{*},$$
$$A^{\mathrm{D}}A(A^{\dagger})^{*}X = A^{\mathrm{D}}A \quad \textit{and} \quad XA^{\mathrm{D}}A(A^{\dagger})^{*} = A^{*}A^{\mathrm{D}}A(A^{\dagger})^{*};$$

(xiii) X *satisfies equations*

$$XAA^{\mathrm{D}} = X \quad \textit{and} \quad XA^{\mathrm{D}}A(A^{\dagger})^{*} = A^{*}A^{\mathrm{D}}A(A^{\dagger})^{*};$$

(xiv) X *satisfies equations*

$$A^{*}AA^{\mathrm{D}}(A^{\dagger})^{*}X = X \quad \textit{and} \quad A^{\mathrm{D}}A(A^{\dagger})^{*}X = A^{\mathrm{D}}A.$$

Beside previous algebraic point of view, the outer-star and star-outer matrices can be presented from a geometrical point of view. We see that both algebraic and geometrical approaches are equivalent by Theorem 6.1.1 and Theorem 6.1.6.

Theorem 6.1.6. *Let* $A \in \mathbb{C}^{m\times n}_{T,S}$.

(a) *The system of conditions*

$$(A^{\dagger})^{*}X = P_{\mathcal{R}((A^{\dagger})^{*}A^{(2)}_{T,S}),\mathcal{N}(A^{(2)}_{T,S}AA^{*})} \quad \textit{and} \quad \mathcal{R}(X) \subseteq T \tag{6.4}$$

is consistent and it has a unique solution $X = A^{(2,*)}_{T,S}$.

(b) *The system of conditions*

$$(A^{\dagger})^{*}X = P_{\mathcal{R}(AA^{(2)}_{T,S}),S} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^{*})$$

is consistent and it has a unique solution $X = A^{(*,2)}_{T,S}$.

Proof. (a) By Lemma 6.1.1(i), we know that $(A^{\dagger})^{*}A^{(2,*)}_{T,S} = P_{\mathcal{R}((A^{\dagger})^{*}A^{(2)}_{T,S}),\mathcal{N}(A^{(2)}_{T,S}AA^{*})}$. The equality $A^{(2,*)}_{T,S} = A^{(2)}_{T,S}AA^{*}$ gives $\mathcal{R}(A^{(2,*)}_{T,S}) \subseteq \mathcal{R}(A^{(2)}_{T,S}) = T$. Hence, (6.4) holds for $X = A^{(2,*)}_{T,S}$.

Assume that two matrices X and X_1 satisfy (6.4). We conclude, from $(A^{\dagger})^{*}(X - X_1) = P_{\mathcal{R}((A^{\dagger})^{*}A^{(2)}_{T,S}),\mathcal{N}(A^{(2)}_{T,S}AA^{*})} - P_{\mathcal{R}((A^{\dagger})^{*}A^{(2)}_{T,S}),\mathcal{N}(A^{(2)}_{T,S}AA^{*})} = 0$, that $\mathcal{R}(X - X_1) \subseteq \mathcal{N}((A^{\dagger})^{*}) = \mathcal{N}(A) \subseteq \mathcal{N}(A^{(2)}_{T,S}A)$. Since $\mathcal{R}(X) \subseteq T = \mathcal{R}(A^{(2)}_{T,S}) = \mathcal{R}(A^{(2)}_{T,S}A)$ and $\mathcal{R}(X_1) \subseteq \mathcal{R}(A^{(2)}_{T,S}A)$, we have $\mathcal{R}(X - X_1) \subseteq \mathcal{R}(A^{(2)}_{T,S}A) \cap \mathcal{N}(A^{(2)}_{T,S}A) = \{0\}$. So, $X = X_1$, which implies that $A^{(2,*)}_{T,S}$ is the unique solution to (6.4).

Similarly, we can verify part (b). $\square$

In the following result, we characterize the Drazin-star and star-Drazin matrices from a geometrical point of view.

Corollary 6.1.4. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\mathrm{ind}(A) = k$.

(a) *The system of conditions*

$$(A^{\dagger})^{*}X = P_{\mathcal{R}((A^{\dagger})^{*}A^{\mathrm{D}}),\mathcal{N}(A^{\mathrm{D}}A^{*})} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^{k})$$

is consistent and it has the unique solution $X = A^{\mathrm{D},*}$.

(b) *The system of conditions*

$$(A^{\dagger})^{*}X = P_{\mathcal{R}(A^{k}),\mathcal{N}(A^{k})} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^{*})$$

is consistent and it has the unique solution $X = A^{*,\mathrm{D}}$.

By Theorem 6.1.2 and Theorem 6.1.4, we observe that the outer-star and star-outer matrices are not inner inverses of $(A^{\dagger})^{*}$. Several necessary and sufficient conditions for the outer-star and star-outer matrices to be inner inverses of $(A^{\dagger})^{*}$, are given now. Also, we present some relations between the outer-star and star-outer matrices and different well known generalized inverses which can be verified easily.

Lemma 6.1.2. *Let* $A \in \mathbb{C}^{m\times n}_{T,S}$. *Then:*

(i) $\mathcal{N}(A) = \mathcal{N}(A^{(2)}_{T,S}A) \Leftrightarrow (A^{\dagger})^{*}A^{(2,*)}_{T,S}(A^{\dagger})^{*} = (A^{\dagger})^{*} \Leftrightarrow A^{\dagger}AA^{(2)}_{T,S}A = A^{\dagger}A \Leftrightarrow AA^{(2)}_{T,S}A = A \Leftrightarrow AA^{(2)}_{T,S}AA^{\dagger} = AA^{\dagger} \Leftrightarrow (A^{\dagger})^{*}A^{(*,2)}_{T,S}(A^{\dagger})^{*} = (A^{\dagger})^{*} \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(AA^{(2)}_{T,S})$;

(ii) $AA^{(2,*)}_{T,S} = AA^{(2)}_{T,S} \Leftrightarrow A^{(2,*)}_{T,S} = A^{(2)}_{T,S}$;

(iii) $A_{T,S}^{(2,*)}A = A_{T,S}^{(2)}A \Leftrightarrow A_{T,S}^{(2,*)} = A_{T,S}^{(2),\dagger}$;

(iv) $A_{T,S}^{(2,*)}A = A^\dagger A \Leftrightarrow A_{T,S}^{(2,*)} = A^\dagger$;

(v) $AA_{T,S}^{(2,*)} = AA^\dagger \Leftrightarrow AA_{T,S}^{(2,*)}A = A$;

(vi) $A_{T,S}^{(2,*)} = A^* \Leftrightarrow A_{T,S}^{(2)}A = A^\dagger A \Leftrightarrow A_{T,S}^{(2),\dagger} = A^\dagger$;

(vii) $A_{T,S}^{(*,2)}A = A_{T,S}^{(2)}A \Leftrightarrow A_{T,S}^{(*,2)} = A_{T,S}^{(2)}$;

(viii) $AA_{T,S}^{(*,2)} = AA_{T,S}^{(2)} \Leftrightarrow A_{T,S}^{(*,2)} = A_{T,S}^{\dagger,(2)}$;

(ix) $AA_{T,S}^{(*,2)} = AA^\dagger \Leftrightarrow A_{T,S}^{(*,2)} = A^\dagger$;

(x) $A_{T,S}^{(*,2)}A = A^\dagger A \Leftrightarrow AA_{T,S}^{(*,2)}A = A$;

(xi) $A_{T,S}^{(*,2)} = A^* \Leftrightarrow AA_{T,S}^{(2)} = AA^\dagger \Leftrightarrow A_{T,S}^{\dagger,(2)} = A^\dagger$;

(x) $A_{T,S}^{(2,*)} = 0 \Leftrightarrow A_{T,S}^{(2)} = 0 \Leftrightarrow A_{T,S}^{(*,2)} = 0$.

We have the next relations between various well-known generalized inverses and the Drazin-star and star-Drazin matrices.

Corollary 6.1.5. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. Then:*

(i) $(A^\dagger)^*A^{\mathrm{D},*}(A^\dagger)^* = (A^\dagger)^*$ *iff* $A^\dagger AA^{\mathrm{D}}A = A^\dagger A$ *iff* $AA^{\mathrm{D}}A = A$ *iff* $AA^{\mathrm{D}}AA^\dagger = AA^\dagger$ *iff* $(A^\dagger)^*A^{*,\mathrm{D}}(A^\dagger)^* = (A^\dagger)^*$;

(ii) $A^kA^{\mathrm{D},*}A^k = A^k$ *iff* $A^kA^*A^k = A^k$ *iff* $A^kA^{*,\mathrm{D}}A^k = A^k$;

(iii) $AA^{\mathrm{D},*} = AA^{\mathrm{D}}$ *iff* $A^{\mathrm{D},*} = A^{\mathrm{D}}$;

(iv) $A^{\mathrm{D},*}A = AA^{\mathrm{D}}$ *iff* $A^{\mathrm{D},*} = A^{\mathrm{D},\dagger}$;

(v) $A^{\mathrm{D},*}A = A^\dagger A$ *iff* $A^{\mathrm{D},*} = A^\dagger$;

(vi) $AA^{\mathrm{D},*} = AA^\dagger$ *iff* $AA^{\mathrm{D},*}A = A$;

(vii) $A^{\mathrm{D},*} = A^*$ *iff* $A^{\mathrm{D},\dagger} = A^\dagger$;

(viii) $A^{*,\mathrm{D}}A = AA^{\mathrm{D}}$ *iff* $A^{*,\mathrm{D}} = A^{\mathrm{D}}$;

(ix) $AA^{*,\mathrm{D}} = AA^{\mathrm{D}}$ *iff* $A^{*,\mathrm{D}} = A^{\dagger,\mathrm{D}}$;

(x) $AA^{*,\mathrm{D}} = AA^\dagger$ *iff* $A^{*,\mathrm{D}} = A^\dagger$;

(xi) $A^{*,\mathrm{D}}A = A^\dagger A$ *iff* $AA^{*,\mathrm{D}}A = A$;

(xii) $A^{*,\mathrm{D}} = A^*$ *iff* $A^{\dagger,\mathrm{D}} = A^\dagger$.

Kurata [71] gave maximal classes of matrices Q and S which are inner inverses of A and for which QAS coincides with the core inverse of A. Maximal classes of matrices determining the DMP inverse can be found in [39]. We can develop maximal classes of complex matrices providing the most general form to represent the outer-star matrix. Remark that Theorem 6.1.7 recovers [95, Theorem 2.5] concerning the Drazin-star matrix.

Theorem 6.1.7. *Let $A \in \mathbb{C}_{T,S}^{m\times n}$ and $Q \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

(i) $A_{T,S}^{(2,*)} = QAA^*$;

(ii) $QA = A_{T,S}^{(2)}A$;

(iii) $AQA = AA_{T,S}^{(2)}A$ *and* $\mathcal{R}(QA) = T$;

(iv) $Q = A_{T,S}^{(2)} + Z(I - AA^\dagger)$, *for arbitrary* $Z \in \mathbb{C}^{n\times m}$.

Proof. (i) $\Rightarrow$ (ii): Since $A_{T,S}^{(2,*)} = QAA^* = A_{T,S}^{(2)}AA^*$, we get

$$QA = QAA^\dagger A = (QAA^*)(A^\dagger)^* = A_{T,S}^{(2,*)}(A^\dagger)^* = A_{T,S}^{(2)}AA^*(A^\dagger)^* = A_{T,S}^{(2)}A.$$

(ii) $\Rightarrow$ (iii): The hypothesis $QA = A_{T,S}^{(2)}A$ implies $AQA = AA_{T,S}^{(2)}A$ and $\mathcal{R}(QA) = \mathcal{R}(A_{T,S}^{(2)}A) = \mathcal{R}(A_{T,S}^{(2)}) = T$.

(iii) $\Rightarrow$ (i): By $\mathcal{R}(QA) = T = \mathcal{R}(A_{T,S}^{(2)}A)$ and $AQA = AA_{T,S}^{(2)}A$, we deduce that $QA = A_{T,S}^{(2)}AQA$ and

$$QAA^* = A_{T,S}^{(2)}(AQA)A^* = (A_{T,S}^{(2)}AA_{T,S}^{(2)})AA^* = A_{T,S}^{(2)}AA^* = A_{T,S}^{(2,*)}.$$

(ii) $\Rightarrow$ (iv): Notice that all solutions of the equation $QA = A^{(2)}_{T,S}A$ are get as a sum of a particular solution of $QA = A^{(2)}_{T,S}A$ and the general solution of the homogeneous equation $QA = 0$. For arbitrary $Z \in \mathbb{C}^{n\times m}$, by [8, p. 52], the general solution of $QA = A^{(2)}_{T,S}A$ is represented as

$$Q = A^{(2)}_{T,S} + Z(I - AA^{\dagger}).$$

(iv) $\Rightarrow$ (i): Because $Q = A^{(2)}_{T,S} + Z(I - AA^{\dagger})$, we have $QAA^* = A^{(2)}_{T,S}AA^* = A^{(2,*)}_{T,S}$. $\square$

Theorem 6.1.7 gives [95, Theorem 2.5] related to the Drazin-star matrix.

Corollary 6.1.6. *Let $A, U \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. Then the following statements are equivalent:*

(i) $A^{\mathrm{D},*} = UAA^*$;

(ii) $UA = A^{\mathrm{D}}A$;

(iii) $AUA = AA^{\mathrm{D}}A$ *and* $\mathcal{R}(UA) = \mathcal{R}(A^k)$;

(iv) $U = A^{\mathrm{D}} + Z(I - AA^{\dagger})$, *for arbitrary* $Z \in \mathbb{C}^{n\times n}$.

Maximal classes for which the expression for star-outer matrix still holds, can be obtained similarly as Theorem 6.1.7.

Theorem 6.1.8. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$ and $Q \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

(i) $A^{(*,2)}_{T,S} = A^*AQ$;

(ii) $AQ = AA^{(2)}_{T,S}$;

(iii) $AQA = AA^{(2)}_{T,S}A$ *and* $N(AQ) = N(A^k)$;

(iv) $Q = A^{(2)}_{T,S} + (I - A^{\dagger}A)Z$, *for arbitrary* $Z \in \mathbb{C}^{n\times n}$.

Corollary 6.1.7. *Let $A, U \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. Then the following statements are equivalent:*

(i) $A^{*,D} = A^*AU$;

(ii) $AU = AA^{\mathrm{D}}$;

(iii) $AUA = AA^{\mathrm{D}}A$ *and* $\mathcal{N}(AU) = \mathcal{N}(A^k)$;

(iv) $U = A^{\mathrm{D}} + (I - A^{\dagger}A)Z$, *for arbitrary* $Z \in \mathbb{C}^{n\times n}$.

6.2 Representations of outer-star and star-outer matrices

In this section, we present various representations of outer-star and star-outer matrices.

Firstly, we investigate expressions for the outer-star and star-outer matrices in terms of corresponding group inverses based on the following result.

Lemma 6.2.1. [161] *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Suppose that $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. If A has an outer inverse $A^{(2)}_{T,S}$, then $\mathrm{ind}(AG) = \mathrm{ind}(GA) = 1$ and*

$$A^{(2)}_{T,S} = (GA)^{\#}G = G(AG)^{\#}.$$

Theorem 6.2.1. *If A and G satisfy the conditions of Lemma 6.2.1, then*

$$A^{(2,*)}_{T,S} = (GA)^{\#}GAA^* = P_{T,\mathcal{N}(GA)}A^*$$

and

$$A^{(*,2)}_{T,S} = A^*AG(AG)^{\#} = A^*P_{\mathcal{R}(AG),S}.$$

Proof. Because $\mathcal{R}(G) = T = \mathcal{R}(A^{(2)}_{T,S})$ and $\mathcal{N}(G) = S = \mathcal{N}(A^{(2)}_{T,S})$, it is well known that $G = A^{(2)}_{T,S}AG$ and $G = GAA^{(2)}_{T,S}$, which imply $\mathcal{N}(AG) = \mathcal{N}(G) = S$ and $\mathcal{R}(GA) = \mathcal{R}(G) = T$. By Lemma 6.2.1 and properties of the group inverse, we obtain

$$A^{(2,*)}_{T,S} = (GA)^{\#}GAA^* = P_{\mathcal{R}(GA),\mathcal{N}(GA)}A^* = P_{T,\mathcal{N}(GA)}A^*$$

and

$$A^{(*,2)}_{T,S} = A^*AG(AG)^{\#} = A^*P_{\mathcal{R}(AG),\mathcal{N}(AG)} = A^*P_{\mathcal{R}(AG),S}.$$

$\square$

Remark that, by [129], if A and G satisfy the conditions of Lemma 6.2.1 and $G = EF$ is a full-rank decomposition of G, then FAE is invertible and $A^{(2)}_{T,S} = E(FAE)^{-1}F = A^{(2)}_{\mathcal{R}(F),\mathcal{N}(E)}$, which implies $A^{(2,*)}_{T,S} = E(FAE)^{-1}FAA^*$ and $A^{(*,2)}_{T,S} = A^*AE(FAE)^{-1}F$.

As a consequence of Theorem 6.2.1, we obtain new representations for the Drazin-star and star-Drazin matrices.

Corollary 6.2.1. *If $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$, then*

$$A^{D,*} = (A^l)^\# A^l A^* = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}A^*$$

and

$$A^{*,D} = A^*A^l(A^l)^\# = A^*P_{\mathcal{R}(A^k),\mathcal{N}(A^k)},$$

for $l \geq k$.

Proof. If $G = A^l$ for $l \geq k$ in Theorem 6.2.1, we get

$$A^{D,*} = (A^{l+1})^\# A^{l+1}A^* = P_{\mathcal{R}(A^{l+1}),\mathcal{N}(A^{l+1})}A^* = P_{\mathcal{R}(A^l),\mathcal{N}(A^l)}A^* = (A^l)^\# A^l A^*$$

and similarly $A^{*,D} = A^*A^l(A^l)^\# = A^*P_{\mathcal{R}(A^l),\mathcal{N}(A^l)}$. $\square$

We establish the general integral representations for outer-star and star-outer matrices in the next results.

Theorem 6.2.2. *If A and G satisfy the conditions of Lemma 6.2.1, then*

$$A^{(2,*)}_{T,S} = \int_0^\infty \exp\big[-G(GAG)^*GAt\big]G(GAG)^*GAA^*\,\mathrm{d}t$$

and

$$A^{(*,2)}_{T,S} = \int_0^\infty A^*A\exp\big[-G(GAG)^*GAt\big]G(GAG)^*G\,\mathrm{d}t.$$

Proof. Using [165, Theorem 2.2], we have

$$A^{(2)}_{T,S} = \int_0^\infty \exp\big[-G(GAG)^*GAt\big]G(GAG)^*G\,\mathrm{d}t$$

The rest follows by the definitions of the (T,S)-outer-star and star-(T,S)-outer matrices. $\square$

Theorem 6.2.3. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(i) *If $G_1 \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G_1) = T$ and $\mathcal{N}(G_1) = \mathcal{N}(A^{(2)}_{T,S}AA^*)$, then*

$$A^{(2,*)}_{T,S} = \int_0^\infty \exp\big[-G_1(G_1(A^\dagger)^*G_1)^*G_1(A^\dagger)^*t\big]G_1(G_1(A^\dagger)^*G_1)^*G_1\,\mathrm{d}t.$$

(ii) *If $G_2 \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G_2) = R(A^*AA^{(2)}_{T,S})$ and $\mathcal{N}(G_2) = S$, then*

$$A^{\dagger,(2)}_{T,S} = \int_0^\infty \exp\big[-G_2(G_2(A^\dagger)^*G_2)^*G_2(A^\dagger)^*t\big]G_2(G_2(A^\dagger)^*G_2)^*G_2\,\mathrm{d}t.$$

Proof. (i) We have that $A^{(2,*)}_{T,S} = [(A^\dagger)^*]^{(2)}_{T,\mathcal{N}(A^{(2)}_{T,S}AA^*)}$ by Lemma 6.1.1(iii). Using [165, Theorem 2.2], we deduce that (i) is satisfied.

In a same way, we check part (ii). $\square$

We also study a limit representation for outer-star and star-outer matrices.

Theorem 6.2.4. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, $B \in \mathbb{C}^{n\times s}$ be of rank s and $C \in \mathbb{C}^{s\times m}$ be of rank s. If $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ exists, then*

$$A^{(2,*)}_{\mathcal{R}(B),\mathcal{N}(C)} = \lim_{t\to 0} B(tI + CAB)^{-1}CAA^*$$

and

$$A^{(*,2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \lim_{t\to 0} A^*AB(tI + CAB)^{-1}C.$$

Proof. According to [76, Theorem 7], notice that

$$A^{(2)}_{T,S} = \lim_{t\to 0} B(tI + CAB)^{-1}C.$$

We can easily complete the proof. $\square$

Corollary 6.2.2. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, $B \in \mathbb{C}^{n\times s}$ be of rank s and $C \in \mathbb{C}^{s\times m}$ be of rank s. Suppose that $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ exists (or $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C)$).*

(i) *If $\operatorname{rank}(CABC) = \operatorname{rank}(CA)$, then*

$$A^{(2,*)}_{\mathcal{R}(B),\mathcal{N}(C)} = BB^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}A^*.$$

(ii) *If $\operatorname{rank}(BCAB) = \operatorname{rank}(AB)$, then*

$$A^{(*,2)}_{\mathcal{R}(B),\mathcal{N}(C)} = A^*C^{(1,2)}_{\mathcal{R}(AB),\mathcal{N}(B)}C.$$

Proof. (i) Since $\operatorname{rank}(CAB) = \operatorname{rank}(C) = \operatorname{rank}(B) = s$, we obtain

$$\operatorname{rank}(CA) \le \operatorname{rank}(C) = \operatorname{rank}(CAB) \le \operatorname{rank}(CA),$$

which gives $\operatorname{rank}(CA) = \operatorname{rank}(C)$. Further, from $\mathcal{R}(CA) \subseteq \mathcal{R}(C)$, we have $\mathcal{R}(CA) = \mathcal{R}(C)$. By $\operatorname{rank}(CABC) = \operatorname{rank}(CA) = \operatorname{rank}(C) = \operatorname{rank}(B)$, we deduce that $B^{(2)}_{\mathcal{R}(CA),\mathcal{N}(CA)} = B^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}$ exists. Following [161], we observe that

$$B^{(2)}_{\mathcal{R}(CA),\mathcal{N}(CA)} = \lim_{t\to 0} (tI + CAB)^{-1} CA.$$

Applying Theorem 6.2.4, it follows

$$A^{(2,*)}_{\mathcal{R}(B),\mathcal{N}(C)} = \lim_{t\to 0} B(tI + CAB)^{-1}CAA^* = BB^{(2)}_{\mathcal{R}(CA),\mathcal{N}(CA)}A^* = BB^{(1,2)}_{\mathcal{R}(C),\mathcal{N}(CA)}A^*.$$

Analogously, we show part (ii). $\square$

Theorem 6.2.5. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(i) *If $B_1 \in \mathbb{C}^{n\times s}_s$ and $C_1 \in \mathbb{C}^{s\times m}_s$ such that $\mathcal{R}(B_1) = T$ and $\mathcal{N}(C_1) = N(A^{(2)}_{T,S}AA^*)$, then*

$$A^{(2,*)}_{T,S} = \lim_{t\to 0} B_1(tI + C_1(A^\dagger)^*B_1)^{-1}C_1.$$

(ii) *If $B_2 \in \mathbb{C}^{n\times s}_s$ and $C_2 \in \mathbb{C}^{s\times m}_s$ such that $\mathcal{R}(B_2) = \mathcal{R}(A^*AA^{(2)}_{T,S})$ and $\mathcal{N}(C_2) = S$, then*

$$A^{(2,*)}_{T,S} = \lim_{t\to 0} B_2(tI + C_2(A^\dagger)^*B_2)^{-1}C_2.$$

Proof. By Lemma 6.1.1 and [76, Theorem 7], the proof is clear. $\square$

We investigate a relation between the outer-star matrix and a corresponding nonsingular bordered matrix.

Theorem 6.2.6. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(I - AA^{(2,*)}_{T,S}) \subseteq \mathcal{R}(U) \subseteq \mathcal{N}(A^{(2,*)}_{T,S}) \qquad \text{and} \qquad T \subseteq \mathcal{N}(V) \subseteq \mathcal{N}(I - A^{(2,*)}_{T,S}A).$$

Then the bordered matrix

$$M = \begin{bmatrix} A & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{(2,*)}_{T,S} & (I - A^{(2,*)}_{T,S}A)V^\dagger \\ U^\dagger(I - AA^{(2,*)}_{T,S}) & -U^\dagger(A - AA^{(2,*)}_{T,S}A)V^\dagger \end{bmatrix}. \tag{6.5}$$

Proof. From $\mathcal{R}(A^{(2,*)}_{T,S}) = T \subseteq \mathcal{N}(V)$, we conclude that $VA^{(2,*)}_{T,S} = 0$. Because $\mathcal{R}(I - AA^{(2,*)}_{T,S}) \subseteq \mathcal{R}(U) = \mathcal{R}(UU^\dagger) = \mathcal{N}(I - UU^\dagger)$, we have $(I - UU^\dagger)(I - AA^{(2,*)}_{T,S}) = 0$ which gives $I - AA^{(2,*)}_{T,S} = UU^\dagger(I - AA^{(2,*)}_{T,S})$. Let N be the right hand side of (6.5). Then

$$\begin{aligned} MN &= \begin{bmatrix} AA^{(2,*)}_{T,S} + UU^\dagger(I - AA^{(2,*)}_{T,S}) & A(I - A^{(2,*)}_{T,S}A)V^\dagger - UU^\dagger(I - AA^{(2,*)}_{T,S})AV^\dagger \\ VA^{(2,*)}_{T,S} & V(I - A^{(2,*)}_{T,S}A)V^\dagger \end{bmatrix} \\ &= \begin{bmatrix} AA^{(2,*)}_{T,S} + I - AA^{(2,*)}_{T,S} & (I - AA^{(2,*)}_{T,S})AV^\dagger - (I - AA^{(2,*)}_{T,S})AV^\dagger \\ 0 & VV^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= I. \end{aligned}$$

Hence, M is nonsingular and $M^{-1} = N$. $\square$

In the special case when $A^{(2,*)}_{T,S} = A^{D,*}$ in Theorem 6.2.6, we obtain a relation between the Drazin-star matrix and a corresponding nonsingular bordered matrix.

Corollary 6.2.3. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(I - AA^{D,*}) \subseteq \mathcal{R}(U) \subseteq \mathcal{N}(A^kA^*) \quad \text{and} \quad \mathcal{R}(A^k) \subseteq \mathcal{N}(V) \subseteq \mathcal{N}(I - A^{D,*}A).$$

Then the bordered matrix

$$M = \begin{bmatrix} A & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{D,*} & (I - A^{D,*}A)V^\dagger \\ U^\dagger(I - AA^{D,*}) & -U^\dagger(A - AA^{D,*}A)V^\dagger \end{bmatrix}.$$

Proof. For $A^{(2,*)}_{T,S} = A^{D,*}$ in Theorem 6.2.6, we get this result by $\mathcal{R}(A^D) = \mathcal{R}(A^k)$ and $\mathcal{N}(A^{D,*}) = \mathcal{N}(A^kA^*)$ [95, Lemma 2.1]. $\square$

Replacing the block A in matrix M of Theorem 6.2.6 by $(A^\dagger)^*$, we can show the next result in a similar manner as Theorem 6.2.6.

Theorem 6.2.7. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(U) = \mathcal{N}(A^{(2,*)}_{T,S}) \quad \text{and} \quad T = \mathcal{N}(V).$$

Then the bordered matrix

$$M = \begin{bmatrix} (A^\dagger)^* & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{(2,*)}_{T,S} & (I - A^{(2)}_{T,S}A)V^\dagger \\ U^\dagger(I - (A^\dagger)^*A^{(2,*)}_{T,S}) & -U^\dagger((A^\dagger)^* - (A^\dagger)^*A^{(2)}_{T,S}A)V^\dagger \end{bmatrix}.$$

As a consequence of Theorem 6.2.7, we can get a corresponding result for the Drazin-star matrix.

Corollary 6.2.4. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(U) = \mathcal{N}(A^kA^*) \quad \text{and} \quad \mathcal{R}(A^k) = \mathcal{N}(V).$$

Then the bordered matrix

$$M = \begin{bmatrix} (A^\dagger)^* & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{D,*} & (I - A^{\mathrm{D}}A)V^\dagger \\ U^\dagger(I - (A^\dagger)^*A^{D,*}) & -U^\dagger((A^\dagger)^* - (A^\dagger)^*A^{\mathrm{D}}A)V^\dagger \end{bmatrix}.$$

Analogously, we can show the following results related to star-outer and star-Drazin matrices.

Theorem 6.2.8. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$.*

(i) *Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(I - AA^{(*,2)}_{T,S}) \subseteq \mathcal{R}(U) \subseteq S \qquad \textit{and} \qquad \mathcal{R}(A^{(2,*)}_{T,S}) \subseteq \mathcal{N}(V) \subseteq \mathcal{N}(I - A^{(*,2)}_{T,S}A).$$

Then the bordered matrix

$$M = \begin{bmatrix} A & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{(*,2)}_{T,S} & (I - A^{(*,2)}_{T,S}A)V^\dagger \\ U^\dagger(I - AA^{(*,2)}_{T,S}) & -U^\dagger(A - AA^{(*,2)}_{T,S}A)V^\dagger \end{bmatrix}.$$

(ii) *Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(U) = S \qquad \textit{and} \qquad \mathcal{R}(A^{(2,*)}_{T,S}) = \mathcal{N}(V).$$

Then the bordered matrix

$$M = \begin{bmatrix} (A^\dagger)^* & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{(*,2)}_{T,S} & (I - A^{(*,2)}_{T,S}(A^\dagger)^*)V^\dagger \\ U^\dagger(I - AA^{(2)}_{T,S}) & -U^\dagger((A^\dagger)^* - AA^{(2)}_{T,S}(A^\dagger)^*)V^\dagger \end{bmatrix}.$$

Corollary 6.2.5. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$.*

(i) *Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(I - AA^{*,D}) \subseteq \mathcal{R}(U) \subseteq \mathcal{N}(A^k) \qquad \textit{and} \qquad \mathcal{R}(A^*A^k) \subseteq \mathcal{N}(V) \subseteq \mathcal{N}(I - A^{*,D}A).$$

Then the bordered matrix

$$M = \begin{bmatrix} A & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{*,D} & (I - A^{*,D}A)V^\dagger \\ U^\dagger(I - AA^{*,D}) & -U^\dagger(A - AA^{*,D}A)V^\dagger \end{bmatrix}.$$

(ii) *Suppose that U and V^* are full column rank matrices such that*

$$\mathcal{R}(U) = \mathcal{N}(A^k) \qquad \textit{and} \qquad \mathcal{R}(A^*A^k) = \mathcal{N}(V).$$

Then the bordered matrix

$$M = \begin{bmatrix} (A^\dagger)^* & U \\ V & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{*,D} & (I - A^{*,D}(A^\dagger)^*)V^\dagger \\ U^\dagger(I - A^{\mathrm{D}}A) & -U^\dagger((A^\dagger)^* - A^{\mathrm{D}}A(A^\dagger)^*)V^\dagger \end{bmatrix}.$$

6.3 Representations of Drazin-star and star-Drazin matrices

Using the Hartwig-Spindelböck decomposition, we present the canonical form of the Drazin-star and star-Drazin matrices. The Hartwig-Spindelböck decomposition [49] of any matrix $A \in \mathbb{C}^{n\times n}$ of rank r is given by

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{6.6}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is a diagonal matrix of the nonzero singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

Theorem 6.3.1. *Let $A \in \mathbb{C}^{n\times n}$ be represented as in* (6.6). *Then*

$$A^{D,*} = U \begin{bmatrix} (\Sigma K)^{\mathrm{D}}\Sigma\Sigma^* & 0 \\ 0 & 0 \end{bmatrix} U^*$$

and

$$A^{*,D} = U \begin{bmatrix} (\Sigma K)^{*,D} & (\Sigma K)^*(\Sigma K)^{\mathrm{D}}\Sigma L \\ (\Sigma L)^*(\Sigma K)^{\mathrm{D}}\Sigma K & (\Sigma L)^*(\Sigma K)^{\mathrm{D}}\Sigma L \end{bmatrix} U^*.$$

Proof. By [86], we have

$$A^{\mathrm{D}} = U \begin{bmatrix} (\Sigma K)^{\mathrm{D}} & [(\Sigma K)^{\mathrm{D}}]^2\Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

Therefore,

$$\begin{aligned} A^{D,*} &= A^{\mathrm{D}}AA^* = U \begin{bmatrix} (\Sigma K)^{\mathrm{D}}\Sigma K & (\Sigma K)^{\mathrm{D}}\Sigma L \\ 0 & 0 \end{bmatrix} U^*U \begin{bmatrix} K^*\Sigma^* & 0 \\ L^*\Sigma^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (\Sigma K)^{\mathrm{D}}\Sigma\Sigma^* & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

The second formula can be verified in a similar manner. □

Applying Theorem 6.3.1, we obtain a necessary and sufficient condition for $A^{D,*}$ to be EP.

Corollary 6.3.1. *Let $A \in \mathbb{C}^{n\times n}$ be represented as in* (6.6). *Then $A^{D,*}$ is EP if and only if $(\Sigma K)^{\mathrm{D}}\Sigma\Sigma^*$ is EP.*

We present the expressions for the Drazin-star and star-Drazin matrices by means of the Schur's triangularization factorization: let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} U^*, \tag{6.7}$$

where A_1 is nonsingular upper-triangular matrix, $rank(A_1) = rank(A^k)$ and A_3 is nilpotent of index k.

Theorem 6.3.2. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. If A is represented as in* (6.7)*, then*

$$A^{D,*} = U \begin{bmatrix} A_1^* + A_1DA_2^* & A_1DA_3^* \\ 0 & 0 \end{bmatrix} U^*$$

and

$$A^{*,D} = U \begin{bmatrix} A_1^* & A_1^*A_1D \\ A_2^* & A_2^*A_1D \end{bmatrix} U^*,$$

where $D = \sum\limits_{i=0}^{k-1} A_1^{i-k-1}A_2A_3^{k-1-i}$.

Proof. According to [35, Theorem 2.5], we have

$$A^{\mathrm{D}} = U \begin{bmatrix} A_1^{-1} & D \\ 0 & 0 \end{bmatrix} U^*, \quad AA^{\mathrm{D}} = U \begin{bmatrix} I & A_1D \\ 0 & 0 \end{bmatrix} U^*$$

and

$$A = U \begin{bmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{bmatrix} U^*,$$

which give this result. □

Using an integral representation for the Drazin inverse of a complex square matrix, which does not require any restriction on its eigenvalues and it is proved in [15], the integral representations for the Drazin-star and star-Drazin matrices are presented now.

Theorem 6.3.3. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k$. Then*

$$A^{D,*} = \int_0^\infty \exp\left[-tA^k(A^{2k+1})^*A^{k+1}\right]A^k(A^{2k+1})^*A^{k+1}A^*\,\mathrm{d}t$$

and

$$A^{*,D} = \int_0^\infty A^*A\exp\left[-tA^k(A^{2k+1})^*A^{k+1}\right]A^k(A^{2k+1})^*A^k\,\mathrm{d}t.$$

Proof. It follows by [15, Theorem 2.1]:

$$A^D = \int_0^\infty \exp\left[-tA^k(A^{2k+1})^*A^{k+1}\right]A^k(A^{2k+1})^*A^k\,\mathrm{dt}.$$

□

We give the limit representations for the Drazin-star and star-Drazin matrices by the limit representation of the Drazin inverse established in [90].

Theorem 6.3.4. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = k \le l$. Then*

$$A^{\mathrm{D},*} = \lim_{\lambda\to 0}(A^{l+1} + \lambda I)^{-1}A^{l+1}A^*$$

and

$$A^{*,\mathrm{D}} = \lim_{\lambda\to 0} A^*A^{l+1}(A^{l+1} + \lambda I)^{-1}.$$

Proof. We show this expressions applying the following limit expression from [90]:

$$A^{\mathrm{D}} = \lim_{\lambda\to 0}(A^{l+1} + \lambda I)^{-1}A^l.$$

□

Also, we develop representations of the Drazin-star and star-Drazin matrices based on the full-rank decomposition of a given matrix.

Lemma 6.3.1. [8] *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $A = B_1G_1$ is a full-rank decomposition and $G_iB_i = B_{i+1}G_{i+1}$ are also full-rank decompositions, $i = 1, 2, \ldots, k-1$. Then the following statements hold:*

(i) *G_kB_k is invertible;*

(ii) *$A^k = B_1B_2\ldots B_kG_k\ldots G_2G_1$;*

(iii) *$A^{\mathrm{D}} = B_1B_2\ldots B_k(G_kB_k)^{-k-1}G_k\ldots G_2G_1$;*

(iv) *$A^\dagger = G_1^*(G_1G_1^*)^{-1}(B_1^*B_1)^{-1}B_1^*$.*

In particular, for $k = 1$, then G_1B_1 is invertible and $A^\# = B_1(G_1B_1)^{-2}G_1$.

Theorem 6.3.5. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 6.3.1. Then*

$$A^{D,*} = B_1B_2\ldots B_k(G_kB_k)^{-k}G_k\ldots G_2G_1G_1^*B_1^*$$

and

$$A^{*,D} = G_1^*B_1^*B_1B_2\ldots B_k(G_kB_k)^{-k}G_k\ldots G_2G_1.$$

Proof. Applying Lemma 6.3.1 and

$$\begin{aligned} A^{\mathrm{D}}A &= B_1B_2\ldots B_k(G_kB_k)^{-k-1}G_k\ldots G_2G_1B_1G_1 \\ &= B_1B_2\ldots B_k(G_kB_k)^{-k-1}G_k\ldots G_2B_2G_2G_1 \\ &= B_1B_2\ldots B_k(G_kB_k)^{-k-1}G_kB_kG_k\ldots G_2G_1 \\ &= B_1B_2\ldots B_k(G_kB_k)^{-k}G_k\ldots G_2G_1 \end{aligned}$$

the proof can be completed. □

6.4 Group-star and star-group matrices

Applying results of previous sections, we introduce and characterize new classes of a square matrix of index one.

In the case that $A^{(2)}_{T,S} = A^\#$ in Theorem 6.1.1, we obtain the following result.

Corollary 6.4.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(A) = 1$.*

(a) *The system of equations*

$$X(A^\dagger)^*X = X, \qquad AX = AA^* \qquad \textit{and} \qquad X(A^\dagger)^* = A^\#A$$

is consistent and its unique solution is $X = A^\#AA^*$.

(b) *The system of equations*

$$X(A^\dagger)^*X = X, \qquad XA = A^*A \qquad \text{and} \qquad (A^\dagger)^*X = A^\# A$$

is consistent and its unique solution is $X = A^*AA^\#$.

Definition 6.4.1. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\operatorname{ind}(A) = 1$.

(a) *The group-star matrix of* A *(or the group-star inverse of* $(A^\dagger)^*$*) is defined as*

$$A^{\#,*} = A^\# AA^*.$$

(b) *The star-group matrix of* A *(or the star-group inverse of* $(A^\dagger)^*$*) is defined as*

$$A^{*,\#} = A^*AA^\#.$$

Remark that, $A^{\#,*}$ is a $\{3\}$-inverse of A and $A^{*,\#}$ is a $\{4\}$-inverse of A. Also, if $A \in \mathbb{C}^{n\times n}$ is EP, then $A^{\#,*} = A^* = A^{*,\#}$.

Lemma 6.4.1. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\operatorname{ind}(A) = 1$. *Then:*

(i) $(A^\dagger)^*A^{\#,*}$ *is the orthogonal projector onto* $\mathcal{R}(A)$*;*

(ii) $A^{\#,*}(A^\dagger)^*$ *is a projector onto* $\mathcal{R}(A)$ *along* $\mathcal{N}(A)$*;*

(iii) $A^{\#,*} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$*;*

(iv) $(A^\dagger)^*A^{*,\#}$ *is a projector onto* $\mathcal{R}(A)$ *along* $\mathcal{N}(A)$*;*

(v) $A^{*,\#}(A^\dagger)^*$ *is the orthogonal onto* $\mathcal{R}(A^*)$*;*

(vi) $A^{*,\#} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}$.

We characterize the group-star matrix in the next consequence of Corollary 6.1.2.

Corollary 6.4.2. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\operatorname{ind}(A) = 1$. *The following statements are equivalent:*

(i) $X \in \mathbb{C}^{n\times n}$ *is the group-star matrix of* A*;*

(ii) X *satisfies equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*,$$
$$AX = AA^* \quad \text{and} \quad X(A^\dagger)^* = A^\# A;$$

(iii) X *satisfies equations*

$$A^\# AX = X \quad \text{and} \quad AX = AA^*;$$

(iv) X *satisfies equations*

$$A^\# AXAA^\dagger = X \quad \text{and} \quad AX(A^\dagger)^* = A;$$

(v) X *satisfies equations*

$$A^\# AX = X \quad \text{and} \quad (A^\dagger)^*X = AA^\dagger;$$

(vi) X *satisfies equations*

$$XAA^\dagger = X \quad \text{and} \quad X(A^\dagger)^* = A^\# A;$$

(vii) X *satisfies equations*

$$XAA^\dagger = X \quad \text{and} \quad X(A^\dagger)^*A = A;$$

(viii) X *satisfies equations*

$$XAA^\dagger = X \quad \text{and} \quad X(A^\dagger)^*A^\dagger = A^{\oplus};$$

(ix) X *satisfies equations*

$$XAA^\dagger = X \quad \text{and} \quad XA = A^\# AA^*A;$$

(x) X *satisfies equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X = AA^\dagger \quad \text{and} \quad X(A^\dagger)^* = A^\# A;$$

(xi) X *satisfies equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*,$$
$$(A^\dagger)^*X = AA^\dagger \quad \text{and} \quad X(A^\dagger)^* = A^\# A.$$

Also, we present some characterizations of the star-group matrix.

Corollary 6.4.3. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A)=1$. The following statements are equivalent:*

(i) *$X \in \mathbb{C}^{n\times n}$ is the star-group matrix of A;*

(ii) *X satisfies equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*,$$
$$XA = A^*A \quad \text{and} \quad (A^\dagger)^*X = A^\# A;$$

(iii) *X satisfies equations*
$$XAA^\# = X \quad \text{and} \quad XA = A^*A;$$

(iv) *X satisfies equations*
$$A^\dagger AXAA^\# = X \quad \text{and} \quad (A^\dagger)^*XA = A;$$

(v) *X satisfies equations*
$$XAA^\# = X \quad \text{and} \quad X(A^\dagger)^* = A^\dagger A;$$

(vi) *X satisfies equations*
$$A^\dagger AX = X \quad \text{and} \quad (A^\dagger)^*X = A^\# A;$$

(vii) *X satisfies equations*
$$A^\dagger AX = X \quad \text{and} \quad A(A^\dagger)^*X = A;$$

(viii) *X satisfies equations*
$$A^\dagger AX = X \quad \text{and} \quad A^\dagger(A^\dagger)^*X = A_{\oplus};$$

(ix) *X satisfies equations*
$$A^\dagger AX = X \quad \text{and} \quad AX = AA^*AA^\#;$$

(x) *X satisfies equations*
$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X = A^\# A \quad \text{and} \quad X(A^\dagger)^* = A^\dagger A;$$

(xi) *X satisfies equations*

$$X(A^\dagger)^*X = X, \quad (A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*,$$
$$(A^\dagger)^*X = A^\# A \quad \text{and} \quad X(A^\dagger)^* = A^\dagger A.$$

Notice that, in general, the Drazin-star and star-Drazin matrices are not inner inverses of $(A^\dagger)^*$, but the group-star and star-group matrices are inner inverses of $(A^\dagger)^*$. Precisely, $A^{\#,*}$ is $\{1,2,3\}$-inverse of $(A^\dagger)^*$ and $A^{*,\#}$ is $\{1,2,4\}$-inverse of $(A^\dagger)^*$.

By Corollary 6.1.4, we give characterizations of the group-star and star-group matrices from a geometrical point of view.

Corollary 6.4.4. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A)=1$.*

(a) *The system of conditions*
$$(A^\dagger)^*X = P_{\mathcal{R}(A)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A)$$
is consistent and it has the unique solution $X = A^{\#,*}$.

(b) *The system of conditions*
$$(A^\dagger)^*X = P_{\mathcal{R}(A),\mathcal{N}(A)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*)$$
is consistent and it has the unique solution $X = A^{*,\#}$.

In the part (i) of the next result, we obtain new characterizations of partial isometries, and in parts (ii) and (vi), we present new characterizations of EP matrices.

Lemma 6.4.2. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A)=1$. Then:*

(i) *$AA^{\#,*}A = A$ iff $AA^*A = A$ iff $AA^{*,\#}A = A$ iff $AA^{\#,*} = AA^\dagger$ iff $A^{*,\#}A = A^\dagger A$;*

(ii) *$AA^{\#,*} = AA^\#$ iff $A^{\#,*} = A^\#$ iff $AA^* = AA^\#$ iff $A^* = A_{\oplus}$ iff A is EP;*

(iii) *$A^{\#,*}A = AA^\#$ iff $A^{\#,*} = A^{\oplus}$;*

(iv) *$A^{\#,*}A = A^\dagger A$ iff $A^{\#,*} = A^\dagger$;*

(v) $A^{\#,*} = A^*$ *iff* $A^{\oplus} = A^\dagger$;

(vi) $A^{*,\#}A = AA^\#$ *iff* $A^{*,\#} = A^\#$ *iff* $A^*A = AA^\#$ *iff* $A^* = A^{\oplus}$ *iff A is EP;*

(vii) $AA^{*,\#} = AA^\#$ *iff* $A^{*,\#} = A_{\oplus}$;

(viii) $AA^{*,\#} = AA^\dagger$ *iff* $A^{*,\#} = A^\dagger$;

(ix) $A^{*,\#} = A^*$ *iff* $A_{\oplus} = A^\dagger$.

Using Corollary 6.1.6 and Corollary 6.1.7, we develop maximal classes of complex matrices for which the expressions of the group-star and star-group matrices are valid.

Corollary 6.4.5. *Let* $A, U \in \mathbb{C}^{n\times n}$ *and* $\mathrm{ind}(A) = 1$.

(a) *Then the following statements are equivalent:*

(i) $A^{\#,*} = UAA^*$;

(ii) $UA = A^\# A$;

(iii) $AUA = A$ *and* $\mathcal{R}(UA) = \mathcal{R}(A)$;

(iv) $U = A^\# + Z(I - AA^\dagger)$, *for arbitrary* $Z \in \mathbb{C}^{n\times n}$.

(b) *Then the following statements are equivalent:*

(i) $A^{*,\#} = A^*AU$;

(ii) $AU = AA^\#$;

(iii) $AUA = A$ *and* $\mathcal{N}(AU) = \mathcal{N}(A)$;

(iv) $U = A^\# + (I - A^\dagger A)Z$, *for arbitrary* $Z \in \mathbb{C}^{n\times n}$.

In the case that a square matrix A of index one is given by (6.6), recall that [2]

$$A^\# = U\begin{bmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{bmatrix}U^*.$$

Thus, we have the following representations of the group-star and star-group matrices.

Corollary 6.4.6. *Let* $A \in \mathbb{C}^{n\times n}$ *be represented as in* (6.6) *and* $\mathrm{ind}(A) = 1$. *Then*

$$A^{\#,*} = U\begin{bmatrix} K^{-1}\Sigma & 0 \\ 0 & 0 \end{bmatrix}U^*$$

and

$$A^{*,\#} = U\begin{bmatrix} (\Sigma K)^* & (\Sigma K)^* K^{-1}L \\ (\Sigma L)^* & (\Sigma L)^* K^{-1}L \end{bmatrix}U^*.$$

Now, we can notice that $A^{\#,*}$ is EP.

Corollary 6.4.7. *Let* $A \in \mathbb{C}^{n\times n}$ *be represented as in* (6.6) *and* $\mathrm{ind}(A) = 1$. *Then* $A^{\#,*}$ *is EP. In addition,*

$$(A^{\#,*})^\dagger = (A^{\#,*})^\# = U\begin{bmatrix} \Sigma^{-1}K & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

We present the integral representations and representations based on the full-rank decomposition for the group-star and star-group matrices.

Corollary 6.4.8. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\mathrm{ind}(A) = 1$.

(a) *Then*

$$A^{\#,*} = \int_0^\infty \exp\left[-tA(A^3)^*A^2\right]A(A^3)^*A^2A^*\,\mathrm{d}t$$

and

$$A^{*,\#} = \int_0^\infty A^*A\exp\left[-tA(A^3)^*A^2\right]A(A^3)^*A\,\mathrm{d}t.$$

(b) *If* $A = B_1G_1$ *is a full-rank decomposition, then*

$$A^{\#,*} = B_1(G_1B_1)^{-1}G_1G_1^*B_1^*$$

and

$$A^{*,\#} = G_1^*B_1^*B_1(G_1B_1)^{-1}G_1.$$

6.5 Applications of outer-star and star-outer matrices

Applying outer-star and star-outer matrices, we will solve some systems of linear equations.

Theorem 6.5.1. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the equation*

$$A^{(2)}_{T,S}Ax = A^{(2,*)}_{T,S}b \tag{6.8}$$

is consistent and its general solution is

$$x = A^{(2,*)}_{T,S}b + (I - A^{(2)}_{T,S}A)y, \tag{6.9}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. If x is represented as in (6.9), then x satisfies (6.8):

$$A^{(2)}_{T,S}Ax = A^{(2)}_{T,S}AA^{(2,*)}_{T,S}b = A^{(2)}_{T,S}AA^{(2)}_{T,S}AA^*b = A^{(2,*)}_{T,S}b.$$

Suppose that x is a solution of (6.8). Then $A^{(2,*)}_{T,S}b = A^{(2)}_{T,S}Ax$ gives

$$x = A^{(2,*)}_{T,S}b + x - A^{(2)}_{T,S}Ax = A^{(2,*)}_{T,S}b + (I - A^{(2)}_{T,S}A)x.$$

Hence, the solution x is of the form (6.9). $\square$

Notice that $A^{D,*}b$ is a solution of the equation $A^lx = A^lA^*b$, $\mathrm{ind}(A) = k \leq l$.

Corollary 6.5.1. *Let $A \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A) = k \leq l$ and $b \in \mathbb{C}^n$. Then the equation*

$$A^lx = A^lA^*b \tag{6.10}$$

is consistent and its general solution is

$$x = A^{D,*}b + (I - A^{\mathrm{D}}A)y, \tag{6.11}$$

for arbitrary $y \in \mathbb{C}^n$.

Similarly, as in Theorem 6.5.1, we can prove the next theorem.

Theorem 6.5.2. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the equation*

$$(A^\dagger)^*x = AA^{(2)}_{T,S}b$$

is consistent and its general solution is

$$x = A^{(*,2)}_{T,S}b + (I - A^\dagger A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

We obtain the following consequence of Theorem 6.5.2 in the case that $b \in \mathcal{R}(AA^{(2)}_{T,S})$.

Corollary 6.5.2. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$. Then the equation*

$$(A^\dagger)^*x = b, \qquad b \in \mathcal{R}(AA^{(2)}_{T,S})$$

is consistent and its general solution is

$$x = A^*b + (I - A^\dagger A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

Also, we can solve the equation $(A^\dagger)^*x = b$ when $b \in \mathcal{R}((A^\dagger)^*A^{(2)}_{T,S})$.

Theorem 6.5.3. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$ and $b \in \mathcal{R}((A^\dagger)^*A^{(2)}_{T,S})$. Then $A^{(2,*)}_{T,S}b$ is the unique solution in T of*

$$(A^\dagger)^*x = b. \tag{6.12}$$

Proof. Because $b \in \mathcal{R}((A^\dagger)^*A^{(2)}_{T,S})$, we have $b = (A^\dagger)^*A^{(2)}_{T,S}z$, for some $z \in \mathbb{C}^m$. If $x = A^{(2,*)}_{T,S}b$, then

$$(A^\dagger)^*x = (A^\dagger)^*A^{(2,*)}_{T,S}b = (A^\dagger)^*A^{(2)}_{T,S}AA^*(A^\dagger)^*A^{(2)}_{T,S}z = (A^\dagger)^*A^{(2)}_{T,S}z = b,$$

that is, x is a solution of the system (6.12).

In order to verify that $x = A^{(2,*)}_{T,S}b$ is the unique solution of (6.12) in T, let $x_1 \in T$ be another solution of (6.12). Thus, $x = x_1$ from

$$x - x_1 \in T \cap \mathcal{N}((A^\dagger)^*) \subseteq \mathcal{R}(A^{(2)}_{T,S}) \cap \mathcal{N}(A^{(2,*)}_{T,S}(A^\dagger)^*) = \mathcal{R}(A^{(2)}_{T,S}A) \cap \mathcal{N}(A^{(2)}_{T,S}A) = \{0\}.$$

$\square$

Corollary 6.5.3. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = k$ and $b \in \mathcal{R}((A^\dagger)^* A^{\mathrm{D}})$. Then $A^{D,*}b$ is the unique solution in $\mathcal{R}(A^k)$ of $(A^\dagger)^* x = b$.*

If $\operatorname{ind}(A) = 1$ in Corollary 6.5.3, we obtain the following result.

Corollary 6.5.4. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = 1$ and $b \in \mathcal{R}(A)$. Then $A^{\#,*}b$ is the unique solution in $\mathcal{R}(A)$ of $(A^\dagger)^* x = b$.*

Similarly as Theorem 6.5.3, we verify the following result.

Theorem 6.5.4. *Let $A \in \mathbb{C}^{m\times n}_{T,S}$ and $b \in \mathcal{R}(AA^{(2)}_{T,S})$. Then $A^* b$ is the unique solution in $\mathcal{R}(A^*)$ of $(A^\dagger)^* x = b$.*

6.6 Summary

Our aim is to introduce the outer-star and star-outer matrices using the outer inverse and conjugate transpose of a given rectangular matrix based on [98]. Thus, we present two new class of rectangular matrices which includes Drazin-star and star-Drazin matrices. We present various characterizations and representations of these new matrices. As applications of outer-star and star-outer matrices, we solve corresponding systems of linear equations. This research enriches previous knowledge about composite outer inverses, Drazin-star and star-Drazin matrices.

For further research, it will be interesting for example to consider generalizations of our results to operators between Hilbert spaces [106] and to tensors [154].

Chapter 7

Minimal rank properties of outer inverses

The main idea of this chapter is to show that some outer inverses with prescribed range and/or null space are solutions to appropriate matrix equations with minimal rank property. In this way, we show that proper outer inverses are solutions to minimization problems in solving some matrix equations with respect to the matrix rank.

The definition of the weak Drazin inverse was presented in [11] as a weakened form of the Drazin inverse. Although a weak Drazin inverse lacks some of the properties of the Drazin inverse, such as being unique, it is still easier to find the weak Drazin inverse than the Drazin inverse. Also, the weak Drazin inverse may be applied instead of the Drazin inverse for example in investigating differential equations or Markov chains as well as in its additional own applications.

Consider a square matrix $A \in \mathbb{C}^{n\times n}$ of index $k = \operatorname{ind}(A)$. Then, a matrix $X \in \mathbb{C}^{n\times n}$ represents [11]:

- a weak Drazin inverse of A when

$$XA^{k+1} = A^k;$$

- a minimal rank weak Drazin inverse of A when

$$XA^{k+1} = A^k \quad \text{and} \quad \operatorname{rank}(X) = \operatorname{rank}(A^{\mathrm{D}});$$

- a commuting weak Drazin inverse of A when

$$XA^{k+1} = A^k \quad \text{and} \quad AX = XA.$$

Recall that, by [11], the Drazin inverse is unique minimal rank commuting weak Drazin inverse. Some characterizations of the minimal rank weak Drazin inverse were given in [174]. Also, it was proved in [174] that many recently defined generalized inverses, are special cases of the minimal rank weak Drazin inverse.

The conditions for solvability of matrix equations and studying their explicit solutions were applied in physics, mechanics, control theory, and many other fields [8, 156]. Motivated by theoretical and applied importance of researches involving solvability of system of equations and forms of their solutions, we continue to study this topic.

The aim of this chapter is to investigate solvability of systems of matrix equations which are weaker than system considering in [11, 174] and to solve some constrained minimizations problems. Main novelty of the chapter is unification of solutions of considered matrix equations with corresponding minimization problems. Consequently, we extend some well-known results and give several new results for the weak Drazin inverse. Also, some characterizations for significant Drazin inverse, group inverse and Moore-Penrose inverse are obtained as consequences.

The detailed explanations of our research goals follow based on [114].

(1) For $X \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$, the first problem, which we consider, is to find equivalent conditions for solvability of the constrained system

$$XAB = B \quad \text{and} \quad \operatorname{rank}(X) = \operatorname{rank}(B). \tag{7.1}$$

We will prove that X is a solution to (7.1) if and only if (iff) $X \in A\{2\}_{\mathcal{R}(B),*}$.

(2) In the case that system (7.1) is consistent, we solve the minimization problem

$$\min\ \operatorname{rank}(X) \quad \text{subject to} \quad XAB = B. \tag{7.2}$$

(3) We investigate solvability of system (7.1) with the additional assumptions. Precisely, we add an additional constraint $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$ or $BAX = B$ or $AX = XA$. A minimal rank outer inverse X with prescribed range $\mathcal{R}(B)$ which commutes with A, will be called a commuting minimal rank outer inverse with prescribed range $\mathcal{R}(B)$.

(4) Suppose that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{l\times m}$. We study the solvability of the system

$$CAX = C \quad \text{and} \quad \operatorname{rank}(X) = \operatorname{rank}(C). \tag{7.3}$$

Since we will show that X is a solution to (7.3) iff $X \in A\{2\}_{*,\mathcal{N}(C)}$, a solution X to (7.3) is called a minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$.

(5) If the system (7.3) is consistent, the minimization problem

$$\min\ \operatorname{rank}(X) \quad \text{subject to} \quad CAX = C \tag{7.4}$$

can be solved.

(6) Special cases of the system (7.3) will be the topic of this research. A minimal rank outer inverse X with prescribed kernel $\mathcal{N}(C)$ which commutes with A, will be called a commuting minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$.

(7) Characterizations for the Drazin inverse, group inverse and Moore-Penrose inverse are obtained applying our results.

(8) The solvability of the system which contains equalities from both systems (7.1) and (7.3) is considered. Precisely, in the case that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{l\times m}$, we study the system

$$XAB = B, \quad CAX = C \quad \text{and} \quad \operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(C). \tag{7.5}$$

We will observe that X is a solution to (7.5) iff $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, and a solution X to (7.5) is called a minimal rank outer inverse with predefined range $\mathcal{R}(B)$ and kernel $\mathcal{N}(C)$. Also, we investigate solvability of the system (7.5) with additional conditions.

7.1 Minimal rank outer inverses with prescribed range

The main goals of this section are to consider solvability of the system (7.1) and the minimization problem (7.2). In the first theorem, we will observe that X presents a solution to (7.1) iff X is an outer inverse of A with the prescribed range $\mathcal{R}(B)$. Also, we give some systems of matrix equations which are equivalent to the system (7.1).

Lemma 7.1.1. **(a)** *If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$, it follows*

$$\text{there exists } X \in \mathbb{C}^{n\times m} \text{ such that } XAB = B \Longleftrightarrow \operatorname{rank}(AB) = \operatorname{rank}(B). \tag{7.6}$$

(b) *For $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{l\times m}$, it follows*

$$\text{there exists } X \in \mathbb{C}^{n\times m} \text{ such that } CAX = C \Longleftrightarrow \operatorname{rank}(CA) = \operatorname{rank}(C). \tag{7.7}$$

Proof. **(a)** The equality $XAB = B$ gives $\operatorname{rank}(B) \le \operatorname{rank}(AB) \le \operatorname{rank}(B)$, i.e. $\operatorname{rank}(B) = \operatorname{rank}(AB)$.

On the other hand, $\operatorname{rank}(B) = \operatorname{rank}(AB) \Longleftrightarrow B(AB)^{(1)}AB = B$ (see, for example [156, P. 33]), implies $XAB = B$ in the case $X = B(AB)^{(1)}$.

(b) This statement can be verified using the conjugate transpose matrices in the part **(a)**. $\square$

Theorem 7.1.1. *Suppose that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times k}$.*
(a) *The subsequent statements are mutually equivalent:*

(i) $XAB = B$ *and* $\operatorname{rank}(X) = \operatorname{rank}(B)$*;*

(ii) $XAB = B$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$*;*

(iii) X *is a solution to (1.5), i.e.,* $X \in A\{2\}_{\mathcal{R}(B),*}$*;*

(iv) $X = BB^{\dagger}X$ *and* $XAB = B$*;*

(v) $XAX = X$, $X = BB^{\dagger}X$ *and* $XAB = B$.

(b) *Additionally,*

$$\begin{aligned}&\min\{\operatorname{rank}(X)|\ XAB=B\}=\operatorname{rank}(B)\\&\{\operatorname{rank}(X)|\ XAB=B\}\subseteq[\operatorname{rank}(B),\operatorname{rank}(X)]\\&\{\operatorname{rank}(X)|\ X\in A\{2\}\wedge XAB=B\}\subseteq[\operatorname{rank}(B),\operatorname{rank}(A)]\end{aligned}\tag{7.8}$$

$$\left\{X\in\mathbb{C}^{n\times m}|\ XAB=B\ \wedge\ \operatorname{rank}(X)=\operatorname{rank}(B)\right\}=A\{2\}_{\mathcal{R}(B),*}\tag{7.9}$$

and

$$\begin{aligned}A\{2\}_{\mathcal{R}(B),*}=\Big\{&X:=B(AB)^\dagger+Y(I-(AB)(AB)^\dagger)|Y\in\mathbb{C}^{n\times m}\\&\wedge XAB=B\wedge\operatorname{rank}(X)=\operatorname{rank}(B)\Big\}.\end{aligned}\tag{7.10}$$

Proof. **(a)** (i) $\Rightarrow$ (ii): From $XAB=B$, it follows $\mathcal{R}(B)\subseteq\mathcal{R}(X)$. Further, $\operatorname{rank}(X)=\operatorname{rank}(B)$ gives $\mathcal{R}(X)=\mathcal{R}(B)$.

(ii) $\Rightarrow$ (iii): The hypothesis $\mathcal{R}(X)=\mathcal{R}(B)$ implies $X=BW_1$ for some $W_1\in\mathbb{C}^{k\times m}$. Then $XAX=XABW_1=BW_1=X$.

(iii) $\Rightarrow$ (iv): Since $\mathcal{R}(X)=\mathcal{R}(B)$ and $XAX=X$, it follows

$$X=BU=BB^\dagger(BU)=BB^\dagger X$$

and

$$B=XV=XA(XV)=XAB,$$

for some $U\in\mathbb{C}^{k\times m}$ and $V\in\mathbb{C}^{m\times k}$.

(iv) $\Rightarrow$ (v): The assumptions $X=BB^\dagger X$ and $XAB=B$ imply

$$XAX=(XAB)B^\dagger X=BB^\dagger X=X.$$

(v) $\Rightarrow$ (i): From $X=BB^\dagger X$ and $XAB=B$, it follows $\operatorname{rank}(X)=\operatorname{rank}(B)$. Further, $XAB=BB^\dagger XAB=BB^\dagger B=B$.

(b) It is straightforward that $XAX=X$ implies $\operatorname{rank}(X)\le\operatorname{rank}(A)$. On the other hand, $XAB=B$ implies $\operatorname{rank}(X)\ge\operatorname{rank}(B)$. So, (7.8) holds.

The set identity (7.9) follows from (i) $\iff$ (iii). Finally, the identity (7.10) follows from the general solution to the matrix equation $XAB=B$ [166]. □

Remark that the assumptions $X=BB^\dagger X$ and $XAB=B$, exploited in Theorem 7.1.1, can be replaced with some of the equivalent conditions presented in Corollary 7.1.1. In this way, we can obtain several matrix equations systems with solutions satisfying $X\in A\{2\}_{\mathcal{R}(B),*}$.

Corollary 7.1.1. [109, Corollary 2.4] *Let $A\in\mathbb{C}^{m\times n}$, $X\in\mathbb{C}^{n\times m}$ and $B\in\mathbb{C}^{n\times k}$.*
(a) *If $XAX=X$, notice that the following statements are equivalent:*

(i) $X=BB^\dagger X$;

(ii) $XA=BB^\dagger XA$;

(iii) $XAA^\dagger=BB^\dagger XAA^\dagger$;

(iv) $XAA^*=BB^\dagger XAA^*$;

(v) $\mathcal{R}(X)\subseteq\mathcal{R}(B)$.

(b) *The following statements are equivalent:*

(i) $XAB=B$;

(ii) $XABB^\dagger=BB^\dagger$;

(iii) $XABB^*=BB^*$;

(iv) $XA(B^\dagger)^*=(B^\dagger)^*$.

Under the hypothesis $XAX=X$, we observe that $XAB=B$ is equivalent to $\mathcal{R}(B)\subseteq\mathcal{R}(X)$.

Proposition 7.1.1. *If $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{n\times k}$, there exists $X\in\mathbb{C}^{n\times m}$ satisfying $XAB=B$ and* $\operatorname{rank}(X)=\operatorname{rank}(B)$ *if and only if* $\operatorname{rank}(AB)=\operatorname{rank}(B)$.

Proof. If there exists X such that $XAB=B$ and $\operatorname{rank}(X)=\operatorname{rank}(B)$, by Lemma 7.1.1, we conclude that $\operatorname{rank}(AB)=\operatorname{rank}(B)$.

The hypothesis $\operatorname{rank}(AB)=\operatorname{rank}(B)$ and [132, Theorem 3] imply that there exists $X\in A\{2\}_{\mathcal{R}(B),*}$. By Theorem 7.1.1, we have $XAB=B$ and $\operatorname{rank}(X)=\operatorname{rank}(B)$. □

Because of (7.8), a solution X to (7.1) is called a minimal rank outer inverse with prescribed range $\mathcal{R}(B)$. Notice that a weak Drazin inverse is a particular solution of (7.1) for $m = n$, $B = A^k$ and $k = \operatorname{ind}(A)$. So, we study solvability of a more general system than the system whose solution is the weak Drazin inverse.

For the particular settings $B = A^k$ in Theorem 7.1.1, we obtain the next result which involves characterizations of the minimal rank weak Drazin inverse.

Corollary 7.1.2 generalizes results from [174], since the statements (i)-(iii) of Corollary 7.1.2 are proposed in [174].

Corollary 7.1.2. *For $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$, the next statements are equivalent:*

(i) $XA^{k+1} = A^k$ *and* $\operatorname{rank}(X) = \operatorname{rank}(A^k)$*;*

(ii) $XA^{k+1} = A^k$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$*;*

(iii) $X \in A\{2\}_{\mathcal{R}(A^k),*}$*;*

(iv) $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$*;*

(v) $XAX = X$, $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$*;*

(vi) *X is a minimal rank weak Drazin inverse of A.*

The assumption $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$ in the system (7.1) reduces the results of Theorem 7.1.1 to the smaller class of inner reflexive inverses if $A\{1,2\}_{\mathcal{R}(B),*}$.

Theorem 7.1.2. *Suppose that $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times k}$.*

(a) *The subsequent statements are mutually equivalent:*

(i) $XAB = B$ *and* $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$*;*

(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{R}(AB) = \mathcal{R}(A)$*;*

(iii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{R}(AB) \supseteq \mathcal{R}(A)$*;*

(iv) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $A = AB(AB)^\dagger A$*;*

(v) $XAX = X$, $AXA = A$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$, *i.e.,* $X \in A\{1,2\}_{\mathcal{R}(B),*}$.

(b) *In addition,*

$$\left\{X \in \mathbb{C}^{n\times m} \mid XAB = B,\ \operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)\right\} = A\{1,2\}_{\mathcal{R}(B),*}. \tag{7.11}$$

Proof. **(a)** (i) $\Rightarrow$ (ii): According to Theorem 7.1.1, $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(B)$. Using [132, Theorem 3], $\operatorname{rank}(AB) = \operatorname{rank}(B) = \operatorname{rank}(A)$. Therefore, the fact $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ gives $\mathcal{R}(AB) = \mathcal{R}(A)$.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): These equivalences are clear.

(ii) $\Rightarrow$ (v): It is clear, by Theorem 7.1.1, that $XAB = B$. For some $V \in \mathbb{C}^{k\times n}$, the assumption $\mathcal{R}(AB) = \mathcal{R}(A)$ implies

$$A = ABV = AX(ABV) = AXA.$$

(v) $\Rightarrow$ (i): From the equalities $XAX = X$ and $AXA = A$, we deduce that $\operatorname{rank}(X) = \operatorname{rank}(A)$. The hypothesis $\mathcal{R}(X) = \mathcal{R}(B)$ yields $\operatorname{rank}(X) = \operatorname{rank}(B)$ and

$$B = XT = XA(XT) = XAB,$$

for some $T \in \mathbb{C}^{m\times k}$.

The proof of the part **(b)** follows from the results of the part **(a)** of this theorem. The matrices X satisfying $XAB = B$, $\operatorname{rank}(X) = \operatorname{rank}(B)$ are outer inverses of rank $\operatorname{rank}(X) = \operatorname{rank}(B) \leq \operatorname{rank}(A)$. In the case $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$, outer inverses become $\{1,2\}$-inverses [132]. Consequently, the matrices X satisfying (7.11) are $\{1,2\}$-inverses of rank $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$. $\square$

Proposition 7.1.2. *If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$, there exists $X \in \mathbb{C}^{n\times m}$ that fulfills $XAB = B$ and $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$ if and only if $\operatorname{rank}(AB) = \operatorname{rank}(B) = \operatorname{rank}(A)$.*

When we add the assumption $AX = XA$ in the system (7.1), we get the following characterizations for a commuting minimal rank outer inverse with prescribed range $\mathcal{R}(B)$.

Theorem 7.1.3. *For $A, X, B \in \mathbb{C}^{n\times n}$, the subsequent statements are equivalent each other:*

(i) $XAB = B$, $\operatorname{rank}(X) = \operatorname{rank}(B)$ *and* $AX = XA$*;*

(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $AX = XA$*;*

(iii) $X^2A = AX^2 = X$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$*;*

(iv) $X^2A = AX^2 = X$*,* $X = BB^{\dagger}X$ *and* $XAB = B$*.*

Proof. (i) $\Leftrightarrow$ (ii): It follows by Theorem 7.1.1.

(ii) $\Rightarrow$ (iii): This implication is evident.

(iii) $\Rightarrow$ (ii): Using $X^2A = AX^2 = X$, we get $AX = AX^2A = XA$. Hence, $X = X^2A = XAX$.

(iv) $\Leftrightarrow$ (iii): Applying Theorem 7.1.1, one can verify this implication.
□

By Theorem 7.1.3, we get the next consequence which contains several characterizations for the Drazin inverse. For $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{ind}(A)$, recall that, by [174, Corollary 2.3], X is a minimal rank weak Drazin inverse of A and $AX = XA$ iff $X = A^{\mathrm{D}}$.

Corollary 7.1.3. *Let* $A, X \in \mathbb{C}^{n\times n}$ *and* $k \in \mathbb{N}$*. The subsequent statements are equivalent each other:*

(i) $XA^{k+1} = A^k$*,* $\mathrm{rank}(X) = \mathrm{rank}(A^k)$ *and* $AX = XA$*;*

(ii) $XAX = X$*,* $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $AX = XA$*;*

(iii) $X^2A = AX^2 = X$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$*;*

(iv) $X^2A = AX^2 = X$*,* $X = A^k(A^k)^{\dagger}X$ *and* $XA^{k+1} = A^k$*;*

(v) $X = A^{\mathrm{D}}$*.*

In the case that the hypothesis $BAX = B$ is added to the system (7.1), we present necessary and sufficient conditions for the solvability of new system. The system $XAB = BAX = B$ was considered in [29] with additional assumptions different from our conditions.

Theorem 7.1.4. *The subsequent statements are equivalent each other for* $A, X, B \in \mathbb{C}^{n\times n}$*:*

(i) $XAB = BAX = B$ *and* $\mathrm{rank}(X) = \mathrm{rank}(B)$*;*

(ii) $XAB = B$*,* $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(B)$*;*

(iii) $XAB = B$*,* $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(B)$*;*

(iv) $XAB = B$*,* $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(X)$*;*

(v) $XAB = B$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(X)$*;*

(vi) $XAX = X$*,* $BAX = B$ *and* $\mathcal{R}(X) = \mathcal{R}(B)$*;*

(vii) $XAX = X$*,* $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(B)$*, i.e.* $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(B)}$*;*

(viii) $XAX = X$*,* $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(B)$*;*

(ix) $XAX = X$*,* $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(X)$*.*

Proof. (i) $\Rightarrow$ (ii): Firstly, $BAX = B$ gives $\mathcal{N}(X) \subseteq \mathcal{N}(B)$. Since $\mathrm{rank}(X) = \mathrm{rank}(B)$, then $\dim \mathcal{N}(X) = n - \mathrm{rank}(X) = n - \mathrm{rank}(B) = \dim \mathcal{N}(B)$. So, $\mathcal{N}(X) = \mathcal{N}(B)$.

(ii) $\Rightarrow$ (iii) and (iv): It is evident.

(iii) $\Rightarrow$ (i): Theorem 7.1.1 and assumptions $XAB = B$ and $\mathcal{R}(X) = \mathcal{R}(B)$ imply $XAX = X$ and $\mathrm{rank}(X) = \mathrm{rank}(B)$. The condition $\mathcal{N}(X) \subseteq \mathcal{N}(B)$ yields, for some $V \in \mathbb{C}^{n\times n}$,

$$B = VX = (VX)AX = BAX.$$

(iv) $\Rightarrow$ (v): This implication is clear.

(v) $\Rightarrow$ (ii): From $XAB = B$, we conclude that $\mathcal{R}(B) \subseteq \mathcal{R}(X)$ and $\mathrm{rank}(B) \leq \mathrm{rank}(X)$. Because $\mathcal{N}(B) \subseteq \mathcal{N}(X)$, we have $X = SB$, for some $S \in \mathbb{C}^{n\times n}$, and so $\mathrm{rank}(X) \leq \mathrm{rank}(B)$. Hence, $\mathrm{rank}(X) = \mathrm{rank}(B)$, which implies $\mathcal{N}(X) = \mathcal{N}(B)$ and $\mathcal{R}(B) = \mathcal{R}(X)$.

The rest follows by Theorem 7.1.1. □

As a consequence of Theorem 7.1.4, we get the following result which involves characterizations of the Drazin inverse.

Corollary 7.1.4. *Let* $A, X \in \mathbb{C}^{n\times n}$ *and* $k \in \mathbb{N}$*. The subsequent statements are mutually equivalent:*

(i) $XA^{k+1} = A^{k+1}X = A^k$ *and* $\mathrm{rank}(X) = \mathrm{rank}(A^k)$*;*

(ii) $XA^{k+1} = A^k$*,* $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$*;*

(iii) $XA^{k+1} = A^k$*,* $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$*;*

(iv) $XA^{k+1} = A^k$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$*;*

(v) $XA^{k+1} = A^k$ *and* $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$*;*

(vi) $XAX = X$, $A^{k+1}X = A^k$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$*;*

(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$, *i.e.* $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} = A^{\mathrm{D}}$*;*

(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$*;*

(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$.

For $k = 1$ in Corollary 7.1.4, we obtain characterizations for the group inverse.

Corollary 7.1.5. *The following statements are equivalent for* $A, X \in \mathbb{C}^{n\times n}$*:*

(i) $XA^2 = A^2X = A$ *and* $\operatorname{rank}(X) = \operatorname{rank}(A)$*;*

(ii) $XA^2 = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) = \mathcal{N}(A)$*;*

(iii) $XA^2 = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A)$*;*

(iv) $XA^2 = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(A) \subseteq \mathcal{N}(X)$*;*

(v) $XA^2 = A$ *and* $\mathcal{N}(A) \subseteq \mathcal{N}(X)$*;*

(vi) $XAX = X$, $A^2X = A$ *and* $\mathcal{R}(X) = \mathcal{R}(A)$*;*

(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) = \mathcal{N}(A)$, *i.e.* $X = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)} = A^{\#}$*;*

(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A)$*;*

(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(A) \subseteq \mathcal{N}(X)$.

Theorem 7.1.4 also implies new characterizations for the Moore-Penrose inverse.

Corollary 7.1.6. *The following statements are equivalent for* $A, X \in \mathbb{C}^{n\times n}$*:*

(i) $XAA^* = A^*AX = A^*$ *and* $\operatorname{rank}(X) = \operatorname{rank}(A^*)$*;*

(ii) $XAA^* = A^*$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$*;*

(iii) $XAA^* = A^*$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^*)$*;*

(iv) $XAA^* = A^*$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$*;*

(v) $XAA^* = A^*$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$*;*

(vi) $XAX = X$, $A^*AX = A^*$ *and* $\mathcal{R}(X) = \mathcal{R}(A^*)$*;*

(vii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$, *i.e.,*
$X = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)} = A^{\dagger}$*;*

(viii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^*)$*;*

(ix) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$.

Example 7.1.1. *Consider the matrices*

$$A = \begin{bmatrix} \epsilon+1 & \epsilon & \epsilon & \epsilon & \epsilon+1 \\ \epsilon & \epsilon-1 & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon+1 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon-1 & \epsilon \\ \epsilon+1 & \epsilon & \epsilon & \epsilon & \epsilon+1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2\epsilon+1 & \epsilon & \epsilon \\ \epsilon & 2\epsilon-1 & \epsilon \\ \epsilon & \epsilon & 2\epsilon+1 \\ \epsilon & \epsilon & \epsilon \\ 3\epsilon & \epsilon & \epsilon \end{bmatrix}.$$

Let us generate the candidate solutions X *in the generic form*

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} \end{bmatrix}, \tag{7.12}$$

where $x_{i,j}$, $i,j = 1,\dots,5$ are unevaluated symbols. The general solution X to $XAB = B$ is the matrix

$$\begin{bmatrix}
x_{1,1} & \frac{2\epsilon^3+\epsilon^2-2\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{1,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-2\epsilon+(6\epsilon+3)x_{1,1}+(6\epsilon+3)x_{1,5}-3}{6\epsilon+4} & \frac{4\epsilon^3-\epsilon^2-2\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{1,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{1,5} \\
x_{2,1} & \frac{-7\epsilon^3+3\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{2,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{2,5}}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{\epsilon+(6\epsilon+3)x_{2,1}+(6\epsilon+3)x_{2,5}}{6\epsilon+4} & \frac{\epsilon\left(12\epsilon^3+3\epsilon^2-6\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{2,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{2,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{2,5} \\
x_{3,1} & \frac{\epsilon(\epsilon+1)^2+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{3,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{3,5}}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{5\epsilon+(6\epsilon+3)x_{3,1}+(6\epsilon+3)x_{3,5}+4}{6\epsilon+4} & \frac{-12\epsilon^4-3\epsilon^3+6\epsilon^2+\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{3,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{3,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{3,5} \\
x_{4,1} & \frac{-\epsilon(\epsilon+1)^2+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{4,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{4,5}}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{\epsilon+(6\epsilon+3)x_{4,1}+(6\epsilon+3)x_{4,5}}{6\epsilon+4} & \frac{\epsilon\left(7\epsilon^2+2\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{4,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{4,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{4,5} \\
x_{5,1} & \frac{\epsilon\left(5\epsilon^2-2\epsilon-3\right)+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{5,1}+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{5,5}}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-5\epsilon+(6\epsilon+3)x_{5,1}+(6\epsilon+3)x_{5,5}}{6\epsilon+4} & \frac{\epsilon\left(\epsilon^2+2\epsilon-3\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{5,1}+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{5,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{5,5}
\end{bmatrix}$$

which satisfies $XAB = B$ but does not satisfy $XAX = X$. Ranks of relevant matrices are equal to

$$\operatorname{rank}(B) = \operatorname{rank}(AB) = 3 < \operatorname{rank}(A) = 4 < \operatorname{rank}(X) = 5.$$

The matrix Z obtained by the replacement $x_{1,5} = x_{2,5} = x_{3,5} = x_{4,5} = x_{5,5} = 0$ in X is equal to

$$\begin{bmatrix}
0 & \frac{2\epsilon^3+\epsilon^2-2\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-2\epsilon+(6\epsilon+3)x_{1,5}-3}{6\epsilon+4} & \frac{4\epsilon^3-\epsilon^2-2\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{1,5}-1}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{1,5} \\
0 & \frac{-7\epsilon^3+3\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{2,5}}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{\epsilon+(6\epsilon+3)x_{2,5}}{6\epsilon+4} & \frac{\epsilon\left(12\epsilon^3+3\epsilon^2-6\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{2,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{2,5} \\
0 & \frac{\epsilon(\epsilon+1)^2+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{3,5}}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{5\epsilon+(6\epsilon+3)x_{3,5}+4}{6\epsilon+4} & \frac{-12\epsilon^4-3\epsilon^3+6\epsilon^2+\epsilon+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{3,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{3,5} \\
0 & \frac{\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{4,5}-\epsilon(\epsilon+1)^2}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{\epsilon+(6\epsilon+3)x_{4,5}}{6\epsilon+4} & \frac{\epsilon\left(7\epsilon^2+2\epsilon-1\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{4,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{4,5} \\
0 & \frac{\epsilon\left(5\epsilon^2-2\epsilon-3\right)+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)x_{5,5}}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{(6\epsilon+3)x_{5,5}-5\epsilon}{6\epsilon+4} & \frac{\epsilon\left(\epsilon^2+2\epsilon-3\right)+\left(-12\epsilon^4-8\epsilon^3+5\epsilon^2+6\epsilon+1\right)x_{5,5}}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & x_{5,5}
\end{bmatrix}$$

and satisfies $\operatorname{rank}(Z) = 4 > \operatorname{rank}(B)$. *Then the matrix equations $ZAB = B$ holds, but $ZAZ = Z$ does not hold.*

Finally, consider the matrix Q obtained by the replacement $x_{1,5} = x_{2,5} = x_{3,5} = x_{4,5} = x_{5,5} = 0$ in Z:

$$Q = \begin{bmatrix}
0 & \frac{2\epsilon^3+\epsilon^2-2\epsilon-1}{2(\epsilon-1)\epsilon(3\epsilon+2)} & \frac{-2\epsilon-3}{6\epsilon+4} & \frac{4\epsilon^3-\epsilon^2-2\epsilon-1}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & 0 \\
0 & \frac{3\epsilon-7\epsilon^3}{2\epsilon\left(3\epsilon^2-\epsilon-2\right)} & \frac{\epsilon}{6\epsilon+4} & \frac{12\epsilon^3+3\epsilon^2-6\epsilon-1}{4(\epsilon-1)\epsilon(3\epsilon+2)} & 0 \\
0 & \frac{(\epsilon+1)^2}{2\left(3\epsilon^2-\epsilon-2\right)} & \frac{5\epsilon+4}{6\epsilon+4} & \frac{-12\epsilon^4-3\epsilon^3+6\epsilon^2+\epsilon}{4(\epsilon-1)\epsilon^2(3\epsilon+2)} & 0 \\
0 & -\frac{(\epsilon+1)^2}{2\left(3\epsilon^2-\epsilon-2\right)} & \frac{\epsilon}{6\epsilon+4} & \frac{7\epsilon^2+2\epsilon-1}{4(\epsilon-1)\epsilon(3\epsilon+2)} & 0 \\
0 & \frac{5\epsilon^2-2\epsilon-3}{2(\epsilon-1)(3\epsilon+2)} & -\frac{5\epsilon}{6\epsilon+4} & \frac{\epsilon^2+2\epsilon-3}{4(\epsilon-1)\epsilon(3\epsilon+2)} & 0
\end{bmatrix}.$$

The matrix Q satisfies $\operatorname{rank}(Q) = 3 = \operatorname{rank}(B)$. *Then both the matrix equations $QAB = B$ and $QAQ = Q$ are satisfied, which is in accordance with the results presented in Theorem 7.1.1.*

Now, let us calculate the matrix $X = BU$, where $U \in \mathbb{C}^{5\times 3}$ is in generic form

$$U = \begin{bmatrix}
u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,5} \\
u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} & u_{2,5} \\
u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} & u_{3,5}
\end{bmatrix}.$$

The set of solutions to $BUAB = B$ with respect to U is given by

$$\begin{bmatrix} u_{1,1} & u_{1,2} & \frac{3\epsilon\left(\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}+1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}-\epsilon\left(6\epsilon^2+9\epsilon+1\right)}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\ u_{2,1} & u_{2,2} & -\frac{3\epsilon(2\epsilon+1)\left((\epsilon-1)u_{2,2}+1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{24\epsilon^3+26\epsilon^2+9\epsilon+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}+1}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\ u_{3,1} & u_{3,2} & \frac{6\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{3,2}\epsilon-3\epsilon-1}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{3,2}-4\epsilon^2(4\epsilon+1)}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \end{bmatrix}.$$

Then the set $A\{2\}_{\mathcal{R}(B),*}$ coincides with the set $Y = BU$ which is given in Appendix A.

The rank identities $\operatorname{rank}(Y) = \operatorname{rank}(B)$ *are satisfied.*

7.2 Minimal rank outer inverses with prescribed kernel

This section is devoted to the solvability of the system (7.3) as well as the minimization problem (7.4). Besides some systems of matrix equations which are equivalent to the system (7.3), we present that X is a solution to the system (7.3) iff X is an outer inverse of A with the prescribed kernel $\mathcal{N}(C)$ in the next theorem.

Theorem 7.2.1. *Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{l\times m}$.*

(a) *The subsequent statements are mutually equivalent:*

(i) $CAX = C$ *and* $\operatorname{rank}(X) = \operatorname{rank}(C)$*;*

(ii) $CAX = C$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$*;*

(iii) *X is a solution to (1.6), i.e., $X \in A\{2\}_{*,\mathcal{N}(C)}$;*

(iv) *$X = XC^\dagger C$ and $CAX = C$;*

(v) *$XAX = X$, $X = XC^\dagger C$ and $CAX = C$.*

(b) *In addition,*

$$\begin{aligned} &\min\{\operatorname{rank}(X)|\ \ CAX = C\} = \operatorname{rank}(C) \\ &\{\operatorname{rank}(X)|\ CAX = C\} \subseteq [\operatorname{rank}(C), \operatorname{rank}(X)] \\ &\{\operatorname{rank}(X)|\ X \in A\{2\} \wedge CAX = C\} \subseteq [\operatorname{rank}(C), \operatorname{rank}(A)] \end{aligned} \tag{7.13}$$

$$\left\{X \in \mathbb{C}^{n\times m}|\ CAX = C\ \wedge\ \operatorname{rank}(X) = \operatorname{rank}(C)\right\} = A\{2\}_{*,\mathcal{N}(C)}. \tag{7.14}$$

and

$$\begin{aligned} A\{2\}_{*,\mathcal{N}(C)} = \Big\{X := (CA)^\dagger C + (I - (CA)^\dagger CA)Y|\ Y \in \mathbb{C}^{n\times m} \\ \wedge\ CAX = C \wedge \operatorname{rank}(X) = \operatorname{rank}(C)\}\,. \end{aligned} \tag{7.15}$$

Proof. (i) $\Rightarrow$ (ii): The hypothesis $CAX = C$ implies $\mathcal{N}(X) \subseteq \mathcal{N}(C)$. Since $\operatorname{rank}(X) = \operatorname{rank}(C)$, we deduce that $\mathcal{N}(X) = \mathcal{N}(C)$.

(ii) $\Rightarrow$ (iii): From $\mathcal{N}(X) = \mathcal{N}(C)$, we have $X = W_2C$ for some $W_2 \in \mathbb{C}^{n\times l}$. Then $XAX = W_2CAX = W_2C = X$.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): These equivalences are clear by [109, Theorem 2.6].

(v) $\Rightarrow$ (i): The assumptions $X = XC^\dagger C$ and $CAX = C$ give $\operatorname{rank}(X) = \operatorname{rank}(C)$. Now, $CAX = CAXC^\dagger C = CC^\dagger C = C$.

The rest of the proof is analogous as the proof of Theorem 7.1.1. $\square$

To get new systems of matrix equations which have an outer inverse of A with the prescribed kernel $\mathcal{N}(C)$ as a solution, we can replace the conditions $X = XC^\dagger C$ and $CAX = C$ of Theorem 7.2.1 with some of the following necessary and sufficient conditions.

Remark 7.2.1. *[109, Remark 2.7] Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{l\times m}$.*

(a) *Under the assumption $XAX = X$, the following statements are equivalent:*

(i) *$X = XC^\dagger C$;*

(ii) *$AX = AXC^\dagger C$;*

(iii) *$A^\dagger AX = A^\dagger AXC^\dagger C$;*

(iv) $A^*AX = A^*AXC^\dagger C$;

(v) $\mathcal{N}(C) \subseteq \mathcal{N}(X)$.

(b) *The following statements are equivalent:*

(i) $CAX = C$;

(ii) $C^\dagger CAX = C^\dagger C$;

(iii) $C^*CAX = C^*C$;

(iv) $(C^\dagger)^*AX = (C^\dagger)^*$.

Under the hypothesis $XAX = X$, we observe that $CAX = C$ is equivalent to $\mathcal{N}(X) \subseteq \mathcal{N}(C)$.

Proposition 7.2.1. *If $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{l\times m}$, there exists $X \in \mathbb{C}^{n\times m}$ satisfying $CAX = C$ and $\mathrm{rank}(X) = \mathrm{rank}(C)$ if and only if $\mathrm{rank}(CA) = \mathrm{rank}(C)$.*

Because of (7.13), a solution X to (7.3) is called a minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$.

Theorem 7.2.1 implies the following result.

Corollary 7.2.1. *The following statements are equivalent for $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$:*

(i) $A^{k+1}X = A^k$ *and* $\mathrm{rank}(X) = \mathrm{rank}(A^k)$;

(ii) $A^{k+1}X = A^k$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;

(iii) $X \in A\{2\}_{*,\mathcal{N}(A^k)}$;

(iv) $X = X(A^k)^\dagger A^k$ *and* $A^{k+1}X = A^k$;

(v) $XAX = X$, $X = X(A^k)^\dagger A^k$ *and* $A^{k+1}X = A^k$;

(vi) *X is a minimal rank weak Drazin inverse of A.*

We now consider the solvability of particular cases of the system (7.3). Firstly, we assume that $\mathrm{rank}(X) = \mathrm{rank}(C) = \mathrm{rank}(A)$ holds in the system (7.3). Notice that the following result can be proved as corresponding results of the previous section.

Theorem 7.2.2. *Consider $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{l\times m}$.*

(a) *The subsequent statements are mutually equivalent:*

(i) $CAX = C$ *and* $\mathrm{rank}(X) = \mathrm{rank}(C) = \mathrm{rank}(A)$;

(ii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{N}(A) = \mathcal{N}(CA)$;

(iii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{N}(CA) \subseteq \mathcal{N}(A)$;

(iv) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $A = A(CA)^\dagger CA$;

(v) $XAX = X$, $AXA = A$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$, *i.e.,* $X \in A\{1,2\}_{*,\mathcal{N}(C)}$.

(b) *In addition,*

$$\{X \in \mathbb{C}^{n\times m} |\ CAX = C,\ \mathrm{rank}(X) = \mathrm{rank}(C) = \mathrm{rank}(A)\} = A\{1,2\}_{*,\mathcal{N}(C)}. \tag{7.16}$$

Proposition 7.2.2. *If $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{l\times m}$, there exists $X \in \mathbb{C}^{n\times m}$ satisfying $CAX = C$ and $\mathrm{rank}(X) = \mathrm{rank}(C) = \mathrm{rank}(A)$ if and only if $\mathrm{rank}(CA) = \mathrm{rank}(C) = \mathrm{rank}(A)$.*

Several characterizations of a commuting minimal rank outer inverse with prescribed kernel $\mathcal{N}(C)$ are proposed in Theorem 7.2.3.

Theorem 7.2.3. *Let $A, X, C \in \mathbb{C}^{n\times n}$. The following statements are mutually equivalent:*

(i) $CAX = C$, $\mathrm{rank}(X) = \mathrm{rank}(C)$ *and* $AX = XA$;

(ii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $AX = XA$;

(iii) $X^2A = AX^2 = X$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;

(iv) $X^2A = AX^2 = X$, $X = XC^\dagger C$ *and* $CAX = C$.

Theorem 7.2.3 gives the next result which gives characterizations of the Drazin inverse.

Corollary 7.2.2. *The subsequent statements are equivalent for $A, X, C \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$:*

(i) $A^{k+1}X = A^k$, $\mathrm{rank}(X) = \mathrm{rank}(A^k)$ *and* $AX = XA$;

(ii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $AX = XA$;

(iii) $X^2A = AX^2 = X$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;

(iv) $X^2A = AX^2 = X$, $X = X(A^k)^\dagger A^k$ and $A^{k+1}X = A^k$;
(v) $X = A^{\mathrm{D}}$.

Taking that $XAC = C$ in the system (7.3), we establish necessary and sufficient conditions for a matrix X to be a solution to new system.

Theorem 7.2.4. *Let $A, X, C \in \mathbb{C}^{n\times n}$. The subsequent statements are equivalent each other:*
(i) $CAX = XAC = C$ *and* $\operatorname{rank}(X) = \operatorname{rank}(C)$;
(ii) $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) = \mathcal{R}(C)$;
(iii) $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(C)$;
(iv) $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(C) \subseteq \mathcal{R}(X)$;
(v) $CAX = C$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(C)$;
(vi) $XAX = X$, $XAC = C$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;
(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) = \mathcal{R}(C)$, *i.e.* $X = A^{(2)}_{\mathcal{R}(C),\mathcal{N}(C)}$;
(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(C)$;
(ix) $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $\mathcal{R}(C) \subseteq \mathcal{R}(X)$.

Consequently, by Theorem 7.2.4, we obtain the next characterizations for the Drazin inverse.

Corollary 7.2.3. *The following statements are equivalent for $A, X \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$:*
(i) $A^{k+1}X = A^k$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) = \mathcal{N}(A^k)$;
(ii) $A^{k+1}X = A^k$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$;
(iii) $A^{k+1}X = A^k$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$;
(iv) $A^{k+1}X = A^k$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$;
(v) $XAX = X$, $XA^{k+1} = A^k$ *and* $\mathcal{N}(X) = \mathcal{N}(A^k)$;
(vi) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$, *i.e.* $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} = A^{\mathrm{D}}$;
(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$;
(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^k)$ *and* $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$.

By Corollary 7.2.3, we can characterize the group inverse in the following way.

Corollary 7.2.4. *The subsequent statements are equivalent for $A, X \in \mathbb{C}^{n\times n}$:*
(i) $A^2X = A$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) = \mathcal{N}(A)$;
(ii) $A^2X = A$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A)$;
(iii) $A^2X = A$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(A) \subseteq \mathcal{R}(X)$;
(iv) $A^2X = A$ *and* $\mathcal{N}(X) \subseteq \mathcal{N}(A)$;
(v) $XAX = X$, $XA^2 = A$ *and* $\mathcal{N}(X) = \mathcal{N}(A)$;
(vi) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) = \mathcal{R}(A)$, *i.e.* $X = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)} = A^\#$;
(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A)$;
(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A)$ *and* $\mathcal{R}(A) \subseteq \mathcal{R}(X)$.

According to Theorem 7.2.4, we have more characterizations for the Moore-Penrose inverse.

Corollary 7.2.5. *The subsequent statements are equivalent for $A, X \in \mathbb{C}^{n\times n}$:*
(i) $A^*AX = A^*$, $\mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) = \mathcal{R}(A^*)$;
(ii) $A^*AX = A^*$, $\mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$;
(iii) $A^*AX = A^*$, $\mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(A^*) \subseteq \mathcal{R}(X)$;
(iv) $A^*AX = A^*$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$;
(v) $XAX = X$, $XAA^* = A^*$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$;
(vi) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) = \mathcal{R}(A^*)$, *i.e.* $X = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)} = A^\dagger$;
(vii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$;
(viii) $XAX = X$, $\mathcal{N}(X) = \mathcal{N}(A^*)$ *and* $\mathcal{R}(A^*) \subseteq \mathcal{R}(X)$.

Example 7.2.1. *Consider the matrix A from Example 7.1.1 and the matrix C of rank 3 defined by*

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

Let us generate the candidate solutions X in the generic form (7.12). The general solution X to $CAX = C$ is equal to

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ -\frac{2\epsilon+(9\epsilon+2)x_{3,1}}{9\epsilon-2} & \frac{-5\epsilon-(9\epsilon+2)x_{3,2}+2}{9\epsilon-2} & \frac{5\epsilon-(9\epsilon+2)x_{3,3}+2}{9\epsilon-2} & -\frac{2\epsilon+(9\epsilon+2)x_{3,1}}{9\epsilon-2}\frac{4\epsilon-(9\epsilon+2)x_{3,4}}{9\epsilon-2} & -\frac{2\epsilon+(9\epsilon+2)x_{3,5}}{9\epsilon-2} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \\ -\frac{3\epsilon+(9\epsilon+4)x_{3,1}}{9\epsilon-2} & \frac{6\epsilon-(9\epsilon+4)x_{3,2}}{9\epsilon-2} & \frac{3\epsilon-(9\epsilon+4)x_{3,3}+4}{9\epsilon-2} & \frac{-3\epsilon-(9\epsilon+4)x_{3,4}+2}{9\epsilon-2} & -\frac{3\epsilon+(9\epsilon+4)x_{3,5}}{9\epsilon-2} \\ \frac{5\epsilon+(2-9\epsilon)x_{1,1}+(9\epsilon-1)x_{3,1}-1}{9\epsilon-2} & \frac{-\epsilon+(2-9\epsilon)x_{1,2}+(9\epsilon-1)x_{3,2}}{9\epsilon-2} & \frac{-8\epsilon+(2-9\epsilon)x_{1,3}+(9\epsilon-1)x_{3,3}+1}{9\epsilon-2} & \frac{-\epsilon+(2-9\epsilon)x_{1,4}+(9\epsilon-1)x_{3,4}}{9\epsilon-2} & \frac{5\epsilon+(2-9\epsilon)x_{1,5}+(9\epsilon-1)x_{3,5}-1}{9\epsilon-2} \end{bmatrix}.$$

The matrix X satisfies $CAX = C$ but does not satisfy $XAX = X$. Ranks of relevant matrices are equal to

$$\operatorname{rank}(C) = \operatorname{rank}(CA) = 3 < \operatorname{rank}(A) = 4 < \operatorname{rank}(X) = 5.$$

The matrix Z obtained by the replacement $x_{1,1} = x_{1,2} = x_{1,3} = x_{1,4} = x_{1,5} = 0$ in X satisfies $\operatorname{rank}(Z) = 4 > \operatorname{rank}(B)$. *Then the matrix equations $ZAB = B$ holds, but $ZAZ = Z$ does not hold.*

Finally, consider the matrix Q obtained by the replacement $x_{3,1} = x_{3,2} = x_{3,3} = x_{3,4} = x_{3,5} = 0$ in Z:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{2\epsilon}{9\epsilon-2} & \frac{2-5\epsilon}{9\epsilon-2} & \frac{5\epsilon+2}{9\epsilon-2} & \frac{4\epsilon}{9\epsilon-2} & -\frac{2\epsilon}{9\epsilon-2} \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{3\epsilon}{9\epsilon-2} & \frac{6\epsilon}{9\epsilon-2} & \frac{3\epsilon+4}{9\epsilon-2} & \frac{2-3\epsilon}{9\epsilon-2} & -\frac{3\epsilon}{9\epsilon-2} \\ \frac{5\epsilon-1}{9\epsilon-2} & -\frac{\epsilon}{9\epsilon-2} & \frac{1-8\epsilon}{9\epsilon-2} & -\frac{\epsilon}{9\epsilon-2} & \frac{5\epsilon-1}{9\epsilon-2} \end{bmatrix}.$$

The matrix Q satisfies $\operatorname{rank}(Q) = 3 = \operatorname{rank}(B)$. *Then both the matrix equations $QAB = B$ and $QAQ = Q$ are satisfied, which is in accordance with the results presented in Theorem 7.2.1.*

Now, let us calculate the matrix $X = UC$, where $U \in \mathbb{C}^{5\times 3}$ is in generic form

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \\ u_{4,1} & u_{4,2} & u_{4,3} \\ u_{5,1} & u_{5,2} & u_{5,3} \end{bmatrix}.$$

The set of solutions to $CAUC = C$ with respect to U is given by

$$\begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & 1-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2} & u_{2,3} \\ \frac{(2-9\epsilon)u_{2,1}-6\epsilon}{9\epsilon+2} & u_{3,2} & \frac{\epsilon+(2-9\epsilon)u_{2,3}+2}{9\epsilon+2} \\ \frac{6\epsilon+(9\epsilon+4)u_{2,1}+2}{9\epsilon+2} & -\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{5\epsilon\ +(9\epsilon+4)u_{2,3}+2}{9\epsilon+2} \\ \frac{-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1}{9\epsilon\ +2} & \frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2}-u_{1,2} & \frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\ \epsilon+2} \end{bmatrix}.$$

Then the set $A\{2\}_{*,\mathcal{N}(C)}$ coincides with the set $Y = UC$ is given in Appendix B. The rank identities $\operatorname{rank}(Y) = \operatorname{rank}(C)$ *are satisfied.*

7.3 Minimal rank outer inverses with prescribed range and kernel

Applying results of Sections 7.1 and 7.2, we are able to characterize solvability of the system (7.5). In particular, by Theorem 7.1.1 and Theorem 7.2.1, the system (7.5) has a solution X iff X is an outer inverse of A with the prescribed range $\mathcal{R}(B)$ and kernel $\mathcal{N}(C)$.

Corollary 7.3.1. *Consider $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{l\times m}$.*
(a) *The subsequent statements are mutually equivalent:*

(i) $XAB = B$, $CAX = C$ *and* $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(C)$;

(ii) $XAB = B$, $CAX = C$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;

(iii) X *is a solution to* (1.7), *i.e.*, $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$;

(iv) $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$;

(v) $XAX = X$, $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$.

(b) *In addition, the system* (7.5) *has the unique solution* $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

Theorem 7.1.2 and Theorem 7.2.2 imply the next characterizations of solution to the special system of the system (7.5) with $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(C) = \operatorname{rank}(A)$.

Corollary 7.3.2. (a) *The subsequent statements are equivalent for* $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{n\times k}$ *and* $C \in \mathbb{C}^{l\times m}$:

(i) $XAB = B$, $CAX = C$ *and* $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(C) = \operatorname{rank}(A)$;

(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$, $\mathcal{R}(A) = \mathcal{R}(AB)$ *and* $\mathcal{N}(A) = \mathcal{N}(CA)$;

(iii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$, $\mathcal{R}(A) \subseteq \mathcal{R}(AB)$ *and* $\mathcal{N}(CA) \subseteq \mathcal{N}(A)$;

(iv) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $A = AB(AB)^{\dagger}A = A(CA)^{\dagger}CA$;

(v) $XAX = X$, $AXA = A$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$, *i.e.*, $X \in A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)}$.

(b) *In addition, the constrained system in* (i) *has the unique solution* $X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

Using Theorem 7.1.3 and Theorem 7.2.3, we characterize the solvability of a new system obtained from the system (7.5) adding an extra condition $AX = XA$.

Corollary 7.3.3. *The subsequent statements are equivalent for* $A, X, B, C \in \mathbb{C}^{n\times n}$:

(i) $XAB = B$, $CAX = C$, $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(C)$ *and* $AX = XA$;

(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$ *and* $AX = XA$;

(iii) $X^2A = AX^2 = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ *and* $\mathcal{N}(X) = \mathcal{N}(C)$;

(iv) $X^2A = AX^2 = X$, $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ *and* $CAX = C$.

Example 7.3.1. *Consider*

$$A = \begin{bmatrix} \frac{1}{\epsilon} & \theta & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & \epsilon^3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Let us generate the possible solutions Q in the generic form

$$Q = \begin{bmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{bmatrix},$$

where $q_{i,j}$, $i,j = 1,\dots,3$ are unevaluated symbols. The general solution Q to the system of matrix equations $QAB = B, CAQ = C$ is equal to

$$Q = \begin{bmatrix} 0 & 0 & \epsilon - \epsilon\theta\, q_{2,3} \\ \frac{1}{\theta} & 0 & x_{2,3} \\ -\frac{1}{\theta^2} & \frac{1}{\theta} & -\frac{q_{2,3}}{\theta} \end{bmatrix}.$$

Ranks of relevant matrices are equal to

$$\operatorname{rank}(B) = \operatorname{rank}(AB) = \operatorname{rank}(C) = \operatorname{rank}(CA) = \operatorname{rank}(A) = 2 < \operatorname{rank}(Q) = 3.$$

Consequently, the system of matrix equations $QAB = B, CAQ = C$ holds, but

$$QAQ = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\theta} & 0 & \frac{1}{\theta} \\ -\frac{1}{\theta^2} & \frac{1}{\theta} & -\frac{1}{\theta^2} \end{bmatrix} \neq Q.$$

The important requirement in Corollary 7.3.1 is $\operatorname{rank}(B) = \operatorname{rank}(C) = \operatorname{rank}(A) = \operatorname{rank}(X)$. *To reduce* $\operatorname{rank}(Q)$ *to* $\operatorname{rank}(A)$ *we use the matrix X obtained by the replacement replacements $q_{2,3} \to 1/\theta$ in Q, which gives*

$$X = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\theta} & 0 & \frac{1}{\theta} \\ -\frac{1}{\theta^2} & \frac{1}{\theta} & -\frac{1}{\theta^2} \end{bmatrix}.$$

All requirements in Corollary 7.3.1 are satisfied and all the matrix equations $XAX = X$, $X = BB^{\dagger}X = XC^{\dagger}C$, $XAB = B$ and $CAX = C$ are fulfilled. Also, the matrix equation $AXA = A$ is satisfied, which means $X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}$.

It is important to mention that $B(CAB)^{\dagger}C$ coincides with X, which is in accordance with the Urquhart representation [150] *and its generalizations from* [**?**].

7.4 Summary

The aim of this chapter is to investigate solvability of systems of constrained matrix equations. Main novelty of the paper is the establishment of correlations between solutions of certain constrained matrix equations with corresponding minimization problems [114]. Some well-known results and several new results for the weak Drazin inverse are obtained in particular cases. Certain characterizations for the Drazin inverse, group inverse and Moore-Penrose inverse are obtained as corollaries.

Implementation of stated research highlights can be summarized as follows.

- Conditions (i)-(vi) in Theorem 7.1.1 are solutions to (7.1), while (7.2) is solved in (7.8) and (7.9).
- Conditions (i)-(vi) in Theorem 7.2.1 are solutions to (7.3), while (7.4) is solved in (7.13) and (7.14).
- The unique solution to (7.5) is $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ and conditions (i)-(vi) in Corollary 7.3.1 are conditions for solvability of (7.5).

Appendix A.

$$
\begin{bmatrix}
(2\epsilon+1)u_{1,1}+\epsilon\left(u_{2,1}+u_{3,1}\right) & (2\epsilon+1)u_{1,2}+\epsilon\left(u_{2,2}+u_{3,2}\right) & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{2,2}\epsilon+3u_{3,2}\epsilon+\left(-12\epsilon^3+9\epsilon+3\right)u_{1,2}+2\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{\left(24\epsilon^5+28\epsilon^4-2\epsilon^3-17\epsilon^2-8\epsilon-1\right)u_{1,2}+\epsilon\left(-2\epsilon\left(2\epsilon^2+\epsilon+1\right)+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{3,2}\right)}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{(2\epsilon+1)\left(3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon u_{1,1}+(2\epsilon-1)u_{2,1}+\epsilon u_{3,1} & \epsilon u_{1,2}+(2\epsilon-1)u_{2,2}+\epsilon u_{3,2} & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2-6\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}\epsilon+3u_{3,2}\epsilon-3\left(4\epsilon^3-4\epsilon^2-\epsilon+1\right)u_{2,2}+2\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{12u_{3,2}\epsilon^5+8u_{3,2}\epsilon^4+26\epsilon^4-5u_{3,2}\epsilon^3+15\epsilon^3-6u_{3,2}\epsilon^2-9\epsilon^2+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}\epsilon-u_{3,2}\epsilon-7\epsilon+\left(24\epsilon^5+4\epsilon^4-18\epsilon^3-7\epsilon^2+4\epsilon+1\right)u_{2,2}-1}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1\right)+(2\epsilon-1)\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon u_{1,1}+\epsilon u_{2,1}+(2\epsilon+1)u_{3,1} & \epsilon u_{1,2}+\epsilon u_{2,2}+(2\epsilon+1)u_{3,2} & -\frac{(\epsilon-1)\left(12u_{3,2}\epsilon^3+3(2\epsilon+1)u_{1,2}\epsilon^2+3(2\epsilon+1)u_{2,2}\epsilon^2+12u_{3,2}\epsilon^2-6\epsilon^2+3u_{3,2}\epsilon-6\epsilon-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{24u_{3,2}\epsilon^5+28u_{3,2}\epsilon^4-14\epsilon^4-2u_{3,2}\epsilon^3-7\epsilon^3-17u_{3,2}\epsilon^2+4\epsilon^2+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}\epsilon+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}\epsilon-8u_{3,2}\epsilon+\epsilon-u_{3,2}}{2\epsilon\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(3\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}-1\right)+\epsilon\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{2,1}-2\epsilon(3\epsilon+2)\left((\epsilon-1)u_{2,2}+1\right)\right)+(2\epsilon+1)\left(\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{3,1}+2\epsilon\left(\epsilon+\left(-3\epsilon^2+\epsilon+2\right)u_{3,2}+1\right)\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon\left(u_{1,1}+u_{2,1}+u_{3,1}\right) & \epsilon\left(u_{1,2}+u_{2,2}+u_{3,2}\right) & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2+3\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}\epsilon+3\left(-2\epsilon^2+\epsilon+1\right)u_{2,2}\epsilon+3u_{3,2}\epsilon-3\epsilon-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{12u_{3,2}\epsilon^4+8u_{3,2}\epsilon^3+2\epsilon^3-5u_{3,2}\epsilon^2+13\epsilon^2-6u_{3,2}\epsilon+8\epsilon+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}-u_{3,2}+1}{2\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(-6u_{2,1}\epsilon^3-6u_{2,2}\epsilon^3-6u_{3,1}\epsilon^3-6u_{3,2}\epsilon^3+3u_{2,1}\epsilon^2+2u_{2,2}\epsilon^2+3u_{3,1}\epsilon^2+2u_{3,2}\epsilon^2-\epsilon^2+2\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+6u_{2,1}\epsilon+4u_{2,2}\epsilon+6u_{3,1}\epsilon+4u_{3,2}\epsilon-2\epsilon+\left(-6\epsilon^3+3\epsilon^2+6\epsilon+1\right)u_{1,1}+u_{2,1}+u_{3,1}-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} \\
\epsilon\left(3u_{1,1}+u_{2,1}+u_{3,1}\right) & \epsilon\left(3u_{1,2}+u_{2,2}+u_{3,2}\right) & \frac{\epsilon\left(-6u_{3,2}\epsilon^3+3u_{3,2}\epsilon^2+9\left(-2\epsilon^2+\epsilon+1\right)u_{1,2}\epsilon+3\left(-2\epsilon^2+\epsilon+1\right)u_{2,2}\epsilon+3u_{3,2}\epsilon+3\epsilon-1\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1} & \frac{12u_{3,2}\epsilon^4+8u_{3,2}\epsilon^3-10\epsilon^3-5u_{3,2}\epsilon^2-5\epsilon^2-6u_{3,2}\epsilon+6\epsilon+3\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{1,2}+\left(12\epsilon^4+8\epsilon^3-5\epsilon^2-6\epsilon-1\right)u_{2,2}-u_{3,2}+1}{2\left(6\epsilon^3-3\epsilon^2-6\epsilon-1\right)} & \frac{\epsilon\left(-6u_{2,1}\epsilon^3-6u_{2,2}\epsilon^3-6u_{3,1}\epsilon^3-6u_{3,2}\epsilon^3+3u_{2,1}\epsilon^2+2u_{2,2}\epsilon^2+3u_{3,1}\epsilon^2+2u_{3,2}\epsilon^2+5\epsilon^2+6\left(-3\epsilon^2+\epsilon+2\right)u_{1,2}\epsilon+6u_{2,1}\epsilon+4u_{2,2}\epsilon+6u_{3,1}\epsilon+4u_{3,2}\epsilon-2\epsilon+\left(-18\epsilon^3+9\epsilon^2+18\epsilon+3\right)u_{1,1}+u_{2,1}+u_{3,1}-3\right)}{6\epsilon^3-3\epsilon^2-6\epsilon-1}
\end{bmatrix}.
$$

Appendix B.

$$
\left[\begin{matrix}
2u_{1,1}+u_{1,2}+u_{1,3} & u_{1,1}+u_{1,3} & u_{1,1}+u_{1,2}+2u_{1,3} & u_{1,1}+u_{1,2}+u_{1,3} & 2u_{1,1}+u_{1,2}+u_{1,3} \\
2u_{2,1}+u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 & u_{2,1}+u_{2,3} & u_{2,1}+2u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 & u_{2,1}+u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 & 2u_{2,1}+u_{2,3}-\frac{(9\epsilon+2)u_{3,2}}{9\epsilon-2}+1 \\
\frac{9u_{3,2}\epsilon-11\epsilon+(4-18\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2u_{3,2}+2}{9\epsilon+2} & \frac{-5\epsilon+(2-9\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2}{9\epsilon+2} & \frac{9u_{3,2}\epsilon-4\epsilon+(2-9\epsilon)u_{2,1}+(4-18\epsilon)u_{2,3}+2u_{3,2}+4}{9\epsilon+2} & \frac{9u_{3,2}\epsilon-5\epsilon+(2-9\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2u_{3,2}+2}{9\epsilon+2} & \frac{9u_{3,2}\epsilon-11\epsilon+(4-18\epsilon)u_{2,1}+(2-9\epsilon)u_{2,3}+2u_{3,2}+2}{9\epsilon+2} \\
\frac{2\left(6\epsilon+(9\epsilon+4)u_{2,1}+2\right)}{9\epsilon+2}+\frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{11\epsilon+(9\epsilon+4)u_{2,1}+(9\epsilon+4)u_{2,3}+4}{9\epsilon+2} & \frac{6\epsilon+(9\epsilon+4)u_{2,1}+2}{9\epsilon+2}+\frac{2\left(5\epsilon+(9\epsilon+4)u_{2,3}+2\right)}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{6\epsilon+(9\epsilon+4)u_{2,1}+2}{9\epsilon+2}+\frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 & \frac{2\left(6\epsilon+(9\epsilon+4)u_{2,1}+2\right)}{9\epsilon+2}+\frac{5\epsilon+(9\epsilon+4)u_{2,3}+2}{9\epsilon+2}-\frac{(9\epsilon+4)u_{3,2}}{9\epsilon-2}-1 \\
-u_{1,2}+\frac{2\left(-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1\right)}{9\epsilon+2}+\frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2} & \frac{-9u_{2,1}\epsilon-9u_{2,3}\epsilon-6\epsilon-(9\epsilon+2)u_{1,1}-(9\epsilon+2)u_{1,3}+u_{2,1}+u_{2,3}+1}{9\epsilon+2} & -u_{1,2}+\frac{-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1}{9\epsilon+2}-\frac{2\left(6\epsilon+(9\epsilon+2)u_{1,3}+(9\epsilon-1)u_{2,3}\right)}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2} & -u_{1,2}+\frac{-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1}{9\epsilon+2}+\frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2} & -u_{1,2}+\frac{2\left(-(9\epsilon+2)u_{1,1}+(1-9\epsilon)u_{2,1}+1\right)}{9\epsilon+2}+\frac{-6\epsilon-(9\epsilon+2)u_{1,3}+(1-9\epsilon)u_{2,3}}{9\epsilon+2}+\frac{(9\epsilon-1)u_{3,2}}{9\epsilon-2}
\end{matrix}\right].
$$

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Team of authors

Predrag Stanimirović
University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia.
Siberian Federal University, 79 Svobodny pr. 660041 Krasnoyarsk, Russia.
E-mail: pecko@pmf.ni.ac.rs

Dijana Mosić
University of Niˇs, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niˇs, Serbia.
E-mail: dijana@pmf.ni.ac.rs

Alena Stupina
Siberian Federal University, 79 Svobodny pr. 660041 Krasnoyarsk, Russia.
E-mail: h677hm@gmail.com

Ratikanta Behera
Department of Computational and Data Sciences, Indian Institute of Science, Bangalore, India.
E-mail: ratikanta@iisc.ac.in

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Stupina Alena,
Behera Ratikanta

SOLVING MATRIX APPROXIMATION PROBLEMS USING GENERALIZED INVERSES

MONOGRAPH

ООО «Научно-издательский центр ИНФРА-М»
127214, Москва, ул. Полярная, д. 31В, стр. 1
Тел.: (495) 280-15-96, 280-33-86. Факс: (495) 280-36-29
E-mail: books@infra-m.ru   http://www.infra-m.ru

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